# Invariant Metrics in Complex Analysis

## & a conjecture of Kobayashi and Lang

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The University of Queensland
Virtual Seminar on Geometry with Symmetries

The results of this talk are based on joint work with James Stanfield[1] (Univ. Münster) and some work in preparation with Frédéric Campana (Univ. Lorraine) and Hervé Gaussier (Institut Fourier, Grenoble).

[1] Broder, K., Stanfield, J., A General Schwarz Lemma for Hermitian Manifolds, arXiv:2309.04636.

# The Unit Disk $\mathbf{D}$

The unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ in the complex plane has a number of remarkable properties:

(1) Every holomorphic map $\mathbf{C} \to \mathbf{D}$ is constant.

(2) There is a complete metric on $\mathbf{D}$ with curvature bounded above by a negative constant.

(3) There is a distance function $\mathfrak{K}_{\mathbf{D}}$ for which the automorphisms of $\mathbf{D}$ are isometries and holomorphic self-maps $f : \mathbf{D} \to \mathbf{D}$ are decreasing in the sense that $f^* \mathfrak{K}_{\mathbf{D}} \leq \mathfrak{K}_{\mathbf{D}}$.

(4) The canonical bundle $K_S = \Lambda_S^{1,0}$ of a curve $S$ universally covered by $\mathbf{D}$ is ample.

Statement (1) is just the classical Liouville theorem from one complex variable. The metric in statement (2) is the Poincaré metric of constant Gauss curvature $-4$. Statement (3) is a consequence of the Schwarz–Pick lemma, where $\mathfrak{K}_{\mathbf{D}}$ is the distance function obtained from integrating the Poincaré metric. Statement (4) is a consequence of the uniformization theorem and Riemann–Roch.

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of degree $d > n+1$.

Each of these properties has been used to define classes of 'hyperbolic' complex manifolds. A complex manifold $X$ is said to be

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We will be exclusively interested in compact Kobayashi hyperbolic manifolds. Hence, a compact complex manifold $X$ is Kobayashi hyperbolic if every holomorphic map $\mathbf{C} \to X$ is constant.

Examples:

- Compact quotients of bounded domains $\Omega \subset \mathbf{C}^n$. In particular, ball quotients $\mathbf{B}^n/\Gamma$.
- A generic smooth hypersurface of degree $d \geq 16n^3(5n+4)$ in $\mathbf{P}^{n+1}$.
  - This lower bound is due to Bérczi–Kirwan (2023); building on the work of Siu (2015), Brotbek (2017), Deng (2017), Demailly (2018), and others.
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The main purpose of this talk is to present the most general evidence for the following conjecture of Kobayashi (1970) and Lang (1986).

Conjecture. Let $X$ be a compact Kobayashi hyperbolic Kähler manifold. Then the canonical bundle $K_X$ is ample. In particular, $X$ is projective and admits a Kähler–Einstein metric g with $\mathrm{Ric}(g) = -g$.

Recall that a complex manifold $X$ is Kähler if it admits a Hermitian metric g such that the 2-form $\omega_g(\cdot, \cdot) := g(J\cdot, \cdot)$ is closed, i.e., $d\omega_g = 0$. This is equivalent to the Levi-Civita connection being compatible with the complex structure J in the sense that ${}^{LC}\nabla J = 0$.

Kähler metrics exist in absurd abundance: The Euclidean metric on $\mathbf{C}^n$, Bergman metric on $\mathbf{B}^n$, and Fubini–Study metric on $\mathbf{P}^n$ are Kähler. Further, since complex submanifolds inherit the Kähler condition, projective and Stein manifolds are Kähler.

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Let $X$ be a complex manifold. Let g be a Hermitian metric, locally described in a coordinate chart $(z_1, ..., z_n)$ by

$$g = \sum_{k,\ell} g_{k\bar{\ell}} dz^k \otimes d\bar{z}^\ell,$$

where $g_{k\bar{\ell}} = g\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_\ell}\right)$ is a Hermitian matrix. The (Chern) curvature tensor of g is the $(0,4)$-tensor whose components are locally given by

$$R_{i\bar{j}k\bar{\ell}} := R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_\ell}\right) = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}_j}.$$

The holomorphic sectional curvature of a Hermitian metric g is defined

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controls the value distribution of holomorphic curves in $X$:

(i) If $\mathrm{HSC}_{\mathrm{g}} \leq -\Lambda_0 < 0$, every holomorphic map $\mathbf{C} \to X$ is constant (i.e., $X$ is Kobayashi hyperbolic).

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The holomorphic sectional curvature is not strong enough to control the Ricci curvature: Hitchin (1976) showed that Hirzebruch surfaces $\mathcal{F}_n := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$ for $n > 1$ admit Kähler metrics with HSC$> 0$, but no Kähler metrics of $\mathrm{Ric} > 0$.

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This makes the following conjecture (a special case of the Kobayashi–Lang conjecture) even more surprising:

Conjecture. (Yau). Let $X$ be a compact Kähler manifold with a Hermitian metric $\mathrm{HSC}_{\hat{g}} < 0$. Then the canonical bundle $K_X$ is ample. In particular, $X$ is projective and admits a Kähler–Einstein metric $g_{KE}$ with $\mathrm{Ric}(g_{KE}) = -g_{KE}$.

The conjecture was verified by Heier–Lu–Wong (2010) for projective threefolds with a Kähler metric $\hat{g}$ satsying $\mathrm{HSC}_{\hat{g}} < 0$. Wu–Yau (2016) developed a general strategy, extending the result to arbitrary dimension. The projective assumption was later relaxed to compact Kähler by Tosatti–Yang (2017).

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The Wu–Yau strategy is to proceed by contradiction, assuming that $K_X$ is not ample. Then produce a sequence of Kähler metrics $\mathrm{g}_\varepsilon$, obtain uniform estimates (independent of $\varepsilon > 0$, and obtain the desired contradiction.

If $\hat{\mathrm{g}}$ is the Kähler metric with $\mathrm{HSC}_{\hat{\mathrm{g}}} \leq -\Lambda_0 < 0$, the core estimate is $\hat{\mathrm{g}} \leq C\mathrm{g}_\varepsilon$, or equivalently,

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The main technique for achieving this estimate is the Schwarz lemma, i.e., an estimate on $|\partial f|^2 = \mathrm{tr}_{\mathrm{g}_\varepsilon}(f^* \hat{\mathrm{g}})$, where $f = \mathrm{id} : (X, \mathrm{g}_\varepsilon) \to (X, \hat{\mathrm{g}})$ is the identity map.

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$$\Delta_{\mathrm{g}} \mathrm{tr}_{\mathrm{g}}(f^* \widehat{\mathrm{g}}) \quad = \quad |\nabla \partial f|^2 + \underbrace{\mathrm{Ric}_{k\bar{\ell}} \mathrm{g}^{k\bar{q}} \mathrm{g}^{p\bar{\ell}} f_p^{\alpha} \overline{f_q^{\beta}} \hat{\mathrm{g}}_{\alpha\bar{\beta}}}_{\text{source curvature term}} - \underbrace{\widehat{\mathrm{R}}_{\alpha\bar{\beta}\gamma\bar{\delta}} \left( \mathrm{g}^{i\bar{j}} f_i^{\alpha} \overline{f_j^{\beta}} \right) \left( \mathrm{g}^{p\bar{q}} f_p^{\gamma} \overline{f_q^{\gamma}} \right)}_{\text{target curvature term}},$$

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In particular, to apply the maximum principle, we want a lower bound on the Ricci curvature of g and an upper bound on the target curvature term of $\hat{\mathrm{g}}$.

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# Two general improvements

Since Lu's calculation in 67, there have been two general improvements:

- Yau (1978) applied his maximum principle to this calculation, which permitted significantly more general source manifolds.
- Royden (1980) showed that the target curvature term is controlled from an upper bound on the holomorphic sectional curvature if the target metric is Kähler. This permits significantly more general target manifolds.

In particular, if the holomorphic sectional curvature of $\widehat{g}$ is bounded above $\mathrm{HSC}_{\widehat{g}} \leq -\Lambda_0 \leq 0$, then

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Theorem. Let $X$ be a compact Kähler manifold with a Kähler metric $\mathrm{HSC}_{\hat{g}} < 0$. Then the canonical bundle $K_X$ is ample.

And our main goal is the following:

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Passing from Kähler to Hermitian is very difficult, in general. Hence, a number of classes of Hermitian metrics, generalizing the Kähler condition have been introduced. A very important class of Hermitian metrics are the pluriclosed metrics, defined by $\partial\bar{\partial}\omega = 0$. Such metrics always exist on a compact complex surface (Gauduchon). The bi-invariant metric on a compact semi-simple Lie group of even rank (endowed with its Samelson complex structure) is pluriclosed.

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Passing from Kähler to Hermitian is very difficult, in general. Hence, a number of classes of Hermitian metrics, generalizing the Kähler condition have been introduced. A very important class of Hermitian metrics are the pluriclosed metrics, defined by $\partial\bar{\partial}\omega = 0$. Such metrics always exist on a compact complex surface (Gauduchon). The bi-invariant metric on a compact semi-simple Lie group of even rank (endowed with its Samelson complex structure) is pluriclosed.

Let us recall the main application of the Royden Schwarz lemma:

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# The Real Bisectional Curvature

In the Wu–Yau theorem, the negatively curved metric is only used to control the target curvature term

$$\widehat{\mathrm{R}}_{\alpha\bar{\beta}\gamma\bar{\delta}} \left( \mathrm{g}^{i\bar{j}} f_i^{\alpha} \overline{f_j^{\beta}} \right) \left( \mathrm{g}^{p\bar{q}} f_p^{\gamma} \overline{f_q^{\gamma}} \right)$$

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Theorem. (Yang–Zheng). Let $X$ be a compact Kähler manifold with a Hermitian metric of $\mathrm{RBC}_{\widehat{\mathrm{g}}} < 0$. Then $X$ has ample canonical bundle.

Because the real bisectional curvature is defined to be the curvature constraint that appears as the target curvature term in the Schwarz lemma, it is far from clear if any improvements can be made.

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But it turns out that the real bisectional curvature is not sharp, and the purpose of the present talk is to exhibit the first general improvement on the Schwarz lemma in the Hermitian category since Royden.

Before stating the main workhorse, let us state the main application; the following most general form of the Kobayashi–Lang conjecture:

<u>Theorem.</u> (B.–Stanfield, 2023). Let $X$ be a compact Kähler manifold with a pluriclosed metric $\widehat{\mathrm{g}}$ of $\mathrm{HSC}_{\widehat{\mathrm{g}}} < 0$. Then the canonical bundle $K_X$ is ample. In particular, $X$ is projective and admits a Kähler–Einstein metric $\mathrm{g}$ with $\mathrm{Ric}(\mathrm{g}) = -\mathrm{g}$.

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The core idea is to pull out the skew-symmetric part of $|\nabla \partial f|^2$. If the target metric $\hat{g}$ is not Kähler, the skew-symmetric part yields torsion terms which temper the real bisectional curvature. Hence, we obtain the following general Schwarz lemma:

Theorem. (B.–Stanfield, 2023). Let $f : (X, g) \to (Y, \widehat{g})$ be a holomorphic map between Hermitian manifolds. Then

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This new curvature condition is intrinsic to the Hermitian structure. Remarkably, if $\hat{g}$ is a pluriclosed metric, then

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