Invariant Metrics in Complex Analysis

& a conjecture of Kobayashi and Lang

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The University of Queensland
The University of Melbourne Topology Seminar

The results of this talk are based on joint work with James Stanfield[1] (Univ. Münster) and some work in preparation with Frédéric Campana (Univ. Lorraine) and Hervé Gaussier (Institut Fourier, Grenoble).

[1] Broder, K., Stanfield, J., A General Schwarz Lemma for Hermitian Manifolds, `arXiv:2309.04636`.

The Unit Disk $\mathbf{D}$

The unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ in the complex plane has a number of remarkable properties:

(1) Every holomorphic map $\mathbf{C} \to \mathbf{D}$ is constant.

(2) There is a complete metric on $\mathbf{D}$ with curvature bounded above by a negative constant.

(3) There is a distance function $\mathfrak{K}_{\mathbf{D}}$ for which the automorphisms of $\mathbf{D}$ are isometries and holomorphic self-maps $f : \mathbf{D} \to \mathbf{D}$ are decreasing in the sense that $f^*\mathfrak{K}_{\mathbf{D}} \leq \mathfrak{K}_{\mathbf{D}}$.

(4) The canonical bundle $K_S = \Lambda_S^{1,0}$ of a curve S universally covered by $\mathbf{D}$ is ample.

Statement (1) is just the classical Liouville theorem from one complex variable. The metric in statement (2) is the Poincaré metric of constant Gauss curvature -4. Statement (3) is a consequence of the Schwarz–Pick lemma, where $\mathfrak{K}_{\mathbf{D}}$ is the distance function obtained from integrating the Poincaré metric. Statement (4) is a consequence of the uniformization theorem and Riemann–Roch.

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The Ahlfors–Schwarz lemma argument does not show that ($\hat{2}$) $\implies$ ($\hat{3}$). Greene–Wu (1979) showed that ($\hat{2}$) $\implies$ ($\hat{3}$) by estimating $\mathfrak{K}_X$ directly. Recall that the canonical bundle K_X of a compact complex manifold X ample if the sections of $K_X^{\otimes \ell}$ furnish an embedding $\Phi : X \to \mathbf{P}^{N_\ell}$. The manifolds in ($\hat{4}$) are all projective with Kähler–Einstein metrics g_{KE} with $\mathrm{Ric}(\mathrm{g}_{\mathrm{KE}}) = -\mathrm{g}_{\mathrm{KE}}$. It is clear that condition ($\hat{4}$) is the weakest. The standard example to bring to mind is the Fermat hypersurface

$$F_d := \{z_0^d + \cdots + z_n^d = 0\} \subset \mathbf{P}^n$$

of degree $d > n + 1$.

Each of these properties has been used to define classes of 'hyperbolic' complex manifolds. A complex manifold X is said to be

($\hat{1}$) Brody hyperbolic if every holomorphic map $\mathbf{C} \to X$ is constant.

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Examples of Kobayashi hyperbolic manifolds

We will be exclusively interested in compact Kobayashi hyperbolic manifolds. Hence, a compact complex manifold X is Kobayashi hyperbolic if every holomorphic map $\mathbf{C} \to X$ is constant.

Examples:

- Compact quotients of bounded domains $\Omega \subset \mathbf{C}^n$. In particular, ball quotients $\mathbf{B}^n/\Gamma$.
- A generic smooth hypersurface of degree $d \geq 16n^3(5n+4)$ in $\mathbf{P}^{n+1}$.
  - This lower bound is due to Bérczi–Kirwan (2023); building on the work of Siu (2015), Brotbek (2017), Deng (2017), Demailly (2018), and others.
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Conjecture. Let X be a compact Kobayashi hyperbolic Kähler manifold. Then the canonical bundle K_X is ample. In particular, X is projective and admits a Kähler–Einstein metric g with $\text{Ric}(g) = -g$.

Recall that a complex manifold X is Kähler if it admits a Hermitian metric g such that the 2-form $\omega_g(\cdot, \cdot) := g(J\cdot, \cdot)$ is closed, i.e., $d\omega_g = 0$. This is equivalent to the Levi-Civita connection being compatible with the complex structure J in the sense that ${}^{LC}\nabla J = 0$.

Kähler metrics exist in absurd abundance: The Euclidean metric on $\mathbf{C}^n$, Bergman metric on $\mathbf{B}^n$, and Fubini–Study metric on $\mathbf{P}^n$ are Kähler. Further, since complex submanifolds inherit the Kähler condition, projective and Stein manifolds are Kähler.

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$$g = \sum_{k,\ell} g_{k\bar{\ell}} dz^k \otimes d\bar{z}^\ell,$$

where $g_{k\bar{\ell}} = g\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_\ell}\right)$ is a Hermitian matrix. The (Chern) curvature tensor of g is the $(0,4)$-tensor whose components are locally given by

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The holomorphic sectional curvature of a Hermitian metric g is defined

$$\mathrm{HSC}_g(\xi) = \frac{1}{|\xi|_g^4} \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} \xi^i \overline{\xi}^j \xi^k \overline{\xi}^\ell,$$

where $\xi = \sum_k \xi^k \frac{\partial}{\partial z_k}$.

Curvature Aspects of Hyperbolicity

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controls the value distribution of holomorphic curves in X:

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(ii) If g is Kähler and $\mathrm{HSC}_{\mathrm{g}} > 0$, then X is rationally connected, i.e., any two points are contained in the image of some holomorphic map $\mathbf{P}^1 \to X$.

The holomorphic sectional curvature is not strong enough to control the Ricci curvature: Hitchin (1976) showed that Hirzebruch surfaces $\mathcal{F}_n := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$ for $n > 1$ admit Kähler metrics with HSC> 0, but no Kähler metrics of $\mathrm{Ric} > 0$.

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This makes the following conjecture (a special case of the Kobayashi–Lang conjecture) even more surprising:

Conjecture. (Yau). Let X be a compact Kähler manifold with a Hermitian metric $\mathrm{HSC}_{\hat{g}} < 0$. Then the canonical bundle K_X is ample. In particular, X is projective and admits a Kähler–Einstein metric g_{KE} with $\mathrm{Ric}(g_{\mathrm{KE}}) = -g_{\mathrm{KE}}$.

The conjecture was verified by Heier–Lu–Wong (2010) for projective threefolds with a Kähler metric $\hat{g}$ satsying $\mathrm{HSC}_{\hat{g}} < 0$. Wu–Yau (2016) developed a general strategy, extending the result to arbitrary dimension. The projective assumption was later relaxed to compact Kähler by Tosatti–Yang (2017).

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The Wu–Yau strategy is to proceed by contradiction, assuming that K_X is not ample. Then produce a sequence of Kähler metrics g_{ε}, obtain uniform estimates (independent of $\varepsilon > 0$, and obtain the desired contradiction.

If $\hat{g}$ is the Kähler metric with $\text{HSC}_{\hat{g}} \leq -\Lambda_0 < 0$, the core estimate is $\hat{g} \leq C g_{\varepsilon}$, or equivalently,

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The main technique for achieving this estimate is the Schwarz lemma, i.e., an estimate on $|\partial f|^2 = \text{tr}_{g_{\varepsilon}}(f^* \hat{g})$, where $f = \text{id} : (X, g_{\varepsilon}) \to (X, \hat{g})$ is the identity map.

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Two general improvements

Since Lu's calculation in 67, there have been two general improvements:

- Yau (1978) applied his maximum principle to this calculation, which permitted significantly more general source manifolds.
- Royden (1980) showed that the target curvature term is controlled from an upper bound on the holomorphic sectional curvature if the target metric is Kähler. This permits significantly more general target manifolds.

In particular, if the holomorphic sectional curvature of $\widehat{g}$ is bounded above $\mathrm{HSC}_{\widehat{g}} \leq -\Lambda_0 \leq 0$, then

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And our main goal is the following:

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The Real Bisectional Curvature

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But it turns out that the real bisectional curvature is not sharp, and the purpose of the present talk is to exhibit the first general improvement on the Schwarz lemma in the Hermitian category since Royden.

Before stating the main workhorse, let us state the main application; the following most general form of the Kobayashi–Lang conjecture:

Theorem. (B.–Stanfield, 2023). Let X be a compact Kähler manifold with a pluriclosed metric $\widehat{\mathrm{g}}$ of $\mathrm{HSC}_{\widehat{\mathrm{g}}} < 0$. Then the canonical bundle K_X is ample. In particular, X is projective and admits a Kähler–Einstein metric g with $\mathrm{Ric}(\mathrm{g}) = -\mathrm{g}$.

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The core idea is to pull out the skew-symmetric part of $|\nabla \partial f|^2$. If the target metric $\hat{g}$ is not Kähler, the skew-symmetric part yields torsion terms which temper the real bisectional curvature. Hence, we obtain the following general Schwarz lemma:

Theorem. (B.–Stanfield, 2023). Let $f : (X, g) \to (Y, \widehat{g})$ be a holomorphic map between Hermitian manifolds. Then

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The new target curvature term is then what we call the tempered real bisectional curvature

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This new curvature condition is intrinsic to the Hermitian structure. Remarkably, if $\hat{g}$ is a pluriclosed metric, then

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