Recent developments concerning the Schwarz Lemma with applications to the Wu–Yau Theorem

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Classical Bochner Technique

Let (M, g) be a compact Riemannian manifold[1]. Let $\alpha \in \Omega^1_M$.

$$\Delta_d \alpha = (dd^* + d^* d)\alpha = \nabla^* \nabla \alpha + \mathrm{Ric}_g(\alpha^\sharp, \cdot).$$

If α is *harmonic*, i.e., $\Delta_d \alpha = 0$, then

$$\Delta_d |\alpha|^2 \quad = \quad |\nabla \alpha|^2 + \mathrm{Ric}_g(\alpha^\sharp, \alpha^\sharp).$$

Theorem. (Bochner). If $\mathrm{Ric}_g > 0$, then $b_1(M) = 0$.

[1] connected and orientable.

Let (X, ω) be a Hermitian manifold[2]. Let $\sigma \in H^0(\mathcal{E})$ be a *holomorphic section* of a *holomorphic vector bundle* $\mathcal{E} \to X$.

We want to compute $\Delta_\omega |\sigma|^2 = \mathrm{tr}_\omega(\sqrt{-1}\partial\bar{\partial}|\sigma|^2)$.

[2]Here, ω is the Hermitian metric, locally described by $\omega =_{\text{loc.}} \sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. We maintain the convention of abusively denoting the metric by a 2–form of type $(1,1)$.

Let $\mathcal{E} \to X$ be a complex vector bundle.

Reminder. A *first-order* $\mathbb{C}$*–linear differential operator* $\bar{\partial}^{\mathcal{E}} : H^0(\mathcal{E}) \to \Omega_X^{0,1} \otimes \mathcal{E}$ is said to be *CR operator* if

$$\bar{\partial}^{\mathcal{E}}(f\sigma) = \bar{\partial}f \otimes \sigma + f\,\bar{\partial}^{\mathcal{E}}\sigma.$$

If, in addition,

$$\bar{\partial}^{\mathcal{E}} \circ \bar{\partial}^{\mathcal{E}} = 0,$$

then we call $\bar{\partial}^{\mathcal{E}}$ a *holomorphic structure*.

Theorem. (Koszul–Malgrange). Let $\mathcal{E}$ be a complex vector bundle. Then $\mathcal{E}$ is a *holomorphic vector bundle* if and only if *$\mathcal{E}$ admits a holomorphic structure $\bar{\partial}^{\mathcal{E}}$*.

If $\bar{\partial}^{\mathcal{E}}$ is a *holomorphic structure* on $\mathcal{E}$, we can complete it to a *Hermitian connection* ∇ in the sense that there is a Hermitian connection ∇ such that

$$\nabla^{0,1} = \bar{\partial}^{\mathcal{E}}.$$

If $\mathcal{E} = T^{1,0}X$, this connection is called the *Chern connection*.

The *Bochner formula* for this connection reads:

$$\Delta_\omega |\sigma|^2 = |\nabla \sigma|^2 - \sqrt{-1}\langle \Theta^{\mathcal{E}} \sigma, \sigma \rangle,$$

where $\Theta^{\mathcal{E}}$ is the curvature of the Hermitian metric on $\mathcal{E}$.

Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a *holomorphic map between complex manifolds.*

We can identify ∂f with a section $\partial f \in H^0(\Omega_X^{1,0} \otimes f^* T^{1,0} Y)$.

Inserting this into the *Bochner formula* yields

$$\Delta_\omega |\partial f|^2 = |\nabla \partial f|^2 - \sqrt{-1} \langle \Theta^{\Omega_X^{1,0} \otimes f^* T^{1,0} Y} \partial f, \partial f \rangle.$$

The curvature *splits additively under tensor products*:

$$\Theta^{\Omega_X^{1,0}\otimes f^*T^{1,0}Y} = \Theta^{\Omega_X^{1,0}} \otimes \mathrm{id} + \mathrm{id} \otimes \Theta^{f^*T^{1,0}Y},$$

inverts additively under dualization:

$$\Theta^{\Omega_X^{1,0}\otimes f^*T^{1,0}Y} = -\Theta^{T^{1,0}X} \otimes \mathrm{id} + \mathrm{id} \otimes \Theta^{f^*T^{1,0}Y},$$

and *commutes with pullback*:

$$\Theta^{\Omega_X^{1,0}\otimes f^*T^{1,0}Y} = -\Theta^{T^{1,0}X} \otimes \mathrm{id} + \mathrm{id} \otimes f^*\Theta^{T^{1,0}Y}$$

The *Bochner formula* therefore yields

$$\begin{aligned}\Delta_{\omega_g} |\partial f|^2 &= |\nabla \partial f|^2 + \mathrm{Ric}_{\omega_g} \otimes \omega_g^\sharp \otimes \omega_g^\sharp \otimes \omega_h \otimes \partial f \otimes \overline{\partial f} \\ &\quad - \mathrm{Rm}_{\omega_h} \otimes \omega_g^\sharp \otimes \partial f \otimes \overline{\partial f} \otimes \omega_g^\sharp \otimes \partial f \otimes \overline{\partial f}.\end{aligned}$$

In local coordinates, we have

$$\Delta_{\omega_g} |\partial f|^2 = |\nabla \partial f|^2 + \underbrace{g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}}}_{\text{Ricci}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta}.$$

Here $f_i^\alpha := \frac{\partial f^\alpha}{\partial z_i}$

Royden showed that if the *target metric is Kähler*[3], the target curvature term can be controlled by the *holomorphic sectional curvature*.

Recall: Let ω is a Kähler metric with underlying complex structure J. The restriction of the *sectional curvature* to the J–invariant 2–planes (i.e., 2–planes of the form $\{u, Ju\}$) defines the *holomorphic sectional curvature*.

In terms of the curvature tensor,

$$\text{HSC}_{\omega}(v) := R(v, \overline{v}, v, \overline{v}).$$

[3]Recall: A Hermitian metric is said to be Kähler if the torsion of the Chern connection vanishes.

The *holomorphic sectional curvature* is very natural to the study of complex geometry:

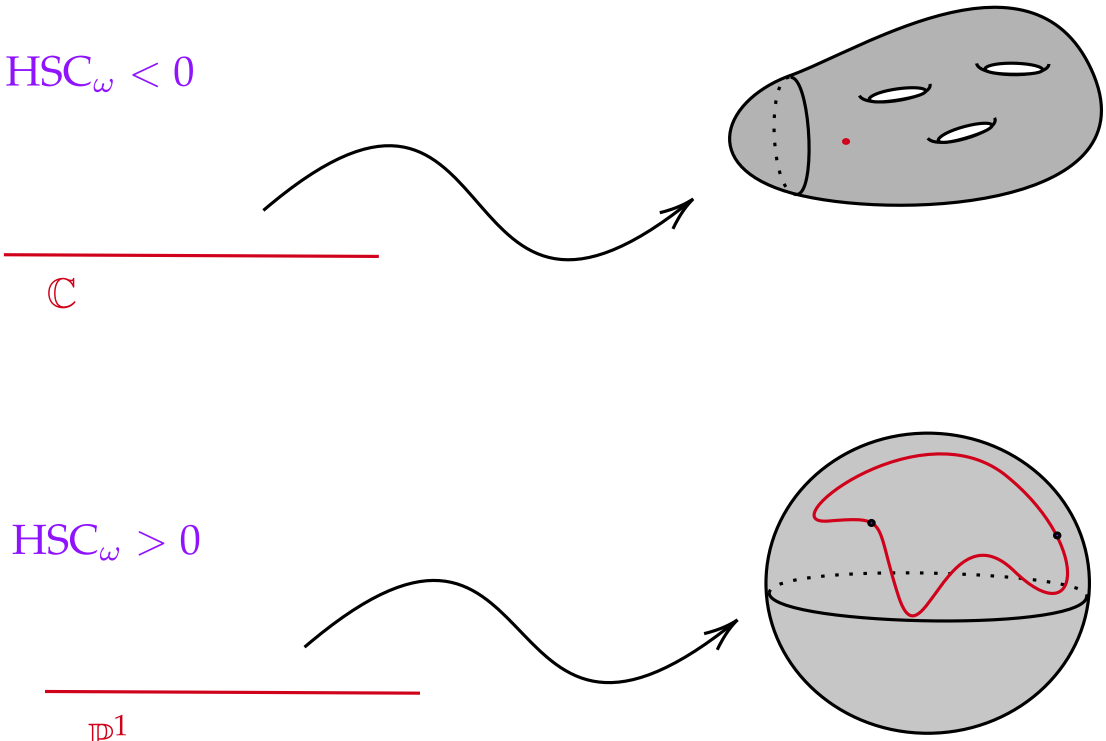
(†) (Ahlfors). $\text{HSC}_\omega < 0 \implies X$ is *Brody hyperbolic*[4]

Every entire curve $\mathbb{C} \to X$ *is constant*.

(†) (Yang). $\text{HSC}_\omega > 0 \implies X$ is *rationally connected*:

Any two points lie in the image of a rational curve $\mathbb{P}^1 \to X$.

[4] If X is compact, this is equivalent to Kobayashi hyperbolicity.


$\mathrm{HSC}_\omega < 0$
$\mathbb{C}$
$\mathrm{HSC}_\omega > 0$
$\mathbb{P}^1$

The argument hinges upon the following polarization argument – called *Royden's trick*:

Proposition. Let $\xi_1, \dots, \xi_\nu$ be ν orthogonal tangent vectors. If $S(\xi, \overline{\eta}, \zeta, \overline{\omega})$ is a *symmetric bi-Hermitian form* in the sense that

(i) $S(\xi, \overline{\eta}, \zeta, \overline{\omega}) = S(\zeta, \overline{\eta}, \xi, \overline{\omega})$,

(ii) $S(\eta, \overline{\xi}, \omega, \overline{\zeta}) = \overline{S}(\xi, \overline{\eta}, \zeta, \overline{\omega})$,

such that for all ξ,

$$S(\xi, \overline{\xi}, \xi, \overline{\xi}) \leq -\kappa_0 \|\xi\|^4,$$

for $\kappa_0 \geq 0$, then

$$\sum_{\alpha,\beta} S(\xi_\alpha, \overline{\xi}_\alpha, \xi_\beta, \overline{\xi}_\beta) \leq -\frac{\nu+1}{2\nu} \kappa_0 \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2.$$

Theorem. (Royden 1980). Let $f : (X, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map between *Kähler manifolds*. Suppose $\mathrm{Ric}_{\omega_g} \geq -C_1 \omega_g$ and $\mathrm{HSC}_{\omega_h} \leq -\kappa_0$ for some constants $C_1, \kappa_0 > 0$. Then

$$\Delta_{\omega_g} \mathrm{tr}_{\omega_g}(f^* \omega_h) \ = \ \Delta_{\omega_g} |\partial f|^2 \ \geq \ -2C_1 + \frac{r+1}{r} \kappa_0 |\partial f|^2,$$

where $r = \mathrm{rank}(\partial f)$.

In particular, if X is compact, then

$$\mathrm{tr}_{\omega_g}(f^* \omega_h) \ = \ |\partial f|^2 \leq \frac{2C_1 r}{(r+1)\kappa_0}.$$

Aside: Classification of Complex Manifolds

The naive approach to *understanding the landscape of complex manifolds* X is to look at *holomorphic functions*

$$X \longrightarrow \mathbb{C}.$$

One runs into trouble quite fast with this approach, however: If X is compact, the *maximum principle forces all such functions to be constant*.

In place of looking at holomorphic maps which take values in the *trivial bundle* $\mathbb{C}$, it is natural to look at holomorphic maps which *takes values in a holomorphic line bundle* $\mathcal{L}$:

$$X \longrightarrow \mathcal{L}.$$

There is only one *line bundle intrinsic to a complex manifold*, the *canonical bundle*

$$K_X := \Lambda_X^{n,0},$$

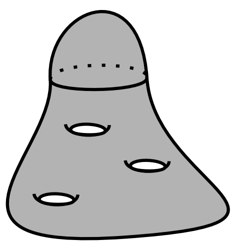
$n = \dim_{\mathbb{C}} X$.

Algebro-Geometric Classification of Complex Manifolds

Understand complex manifolds by means of the *existence/abundance of sections of the canonical bundle* $K_X = \Lambda_X^{n,0}$.

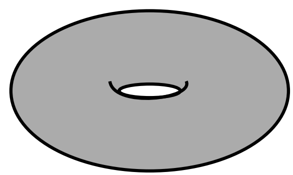
K_X ample

(general type)



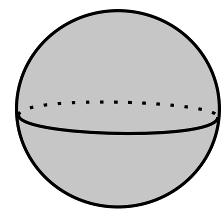
K_X trivial

(Calabi–Yau)



K_X^{-1} ample

(Fano)



Complex-Analytic Classification of Complex Manifolds

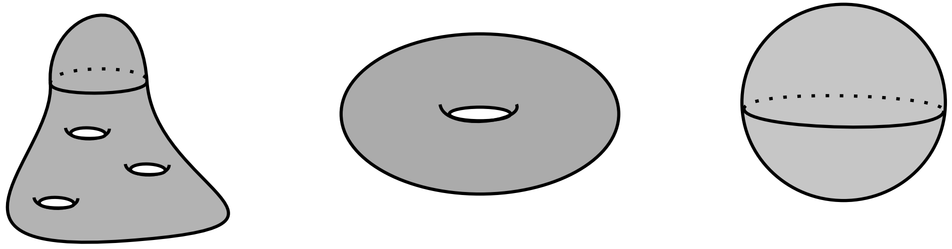
Understand complex manifolds by means of *holomorphic curves* $\mathbb{C} \to X$ and *functions* $X \to \mathbb{C}$:

Lots of holomorphic functions $X \to \mathbb{C}$ Stein manifolds	Lots of holomorphic curves $\mathbb{C} \to X$ Oka/Special manifolds
No holomorphic functions $X \to \mathbb{C}$ Too large	No holomorphic curves $\mathbb{C} \to X$ Kobayashi/Brody hyperbolic manifolds

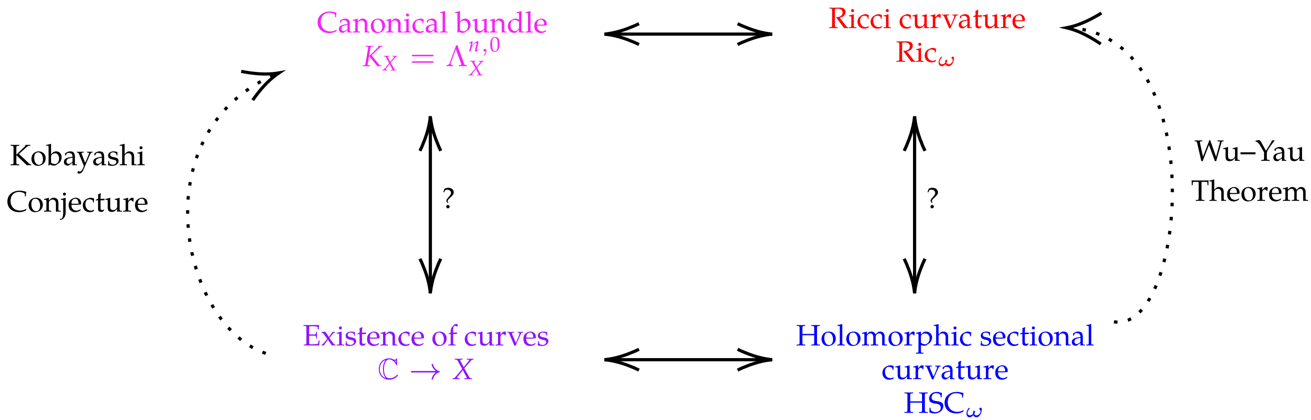
Curvature characterization of Complex Manifolds

Understand complex manifolds by means of *metrics with certain curvature properties*:

$\mathrm{Ric}_\omega < 0$ (general type) | $\mathrm{Ric}_\omega = 0$ (Calabi–Yau) | $\mathrm{Ric}_\omega > 0$ (Fano)



We don't want to simply understand these *distinct means of classification* independently, we want to understand *how they are related*:



The following result is due to Wong (surfaces), Heier–Lu–Wong (projective threefolds), Wu–Yau (projective), Tosatti–Yang (Kähler):[5]

Theorem. Let (X, ω) be a compact Kähler manifold with $\text{HSC}_{\omega} < 0$. Then the canonical bundle K_X is ample.

In particular, we see that

$$\text{HSC}_{\omega} < 0 \implies \exists\, \omega_{\varphi} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi \text{ such that } \text{Ric}_{\omega_{\varphi}} < 0.$$

[5] Recall: A line bundle $\mathcal{L}$ is ample if the sections of $\mathcal{L}^{\otimes k}$ (k large) furnish a holomorphic embedding $\Phi : X \longrightarrow \mathbb{P}^{N_k}$.
In particular, K_X^{-1} is ample if and only if $\text{Ric}_{\omega} > 0$.

The Wu–Yau theorem is an important step towards the more general *Kobayashi conjecture:*

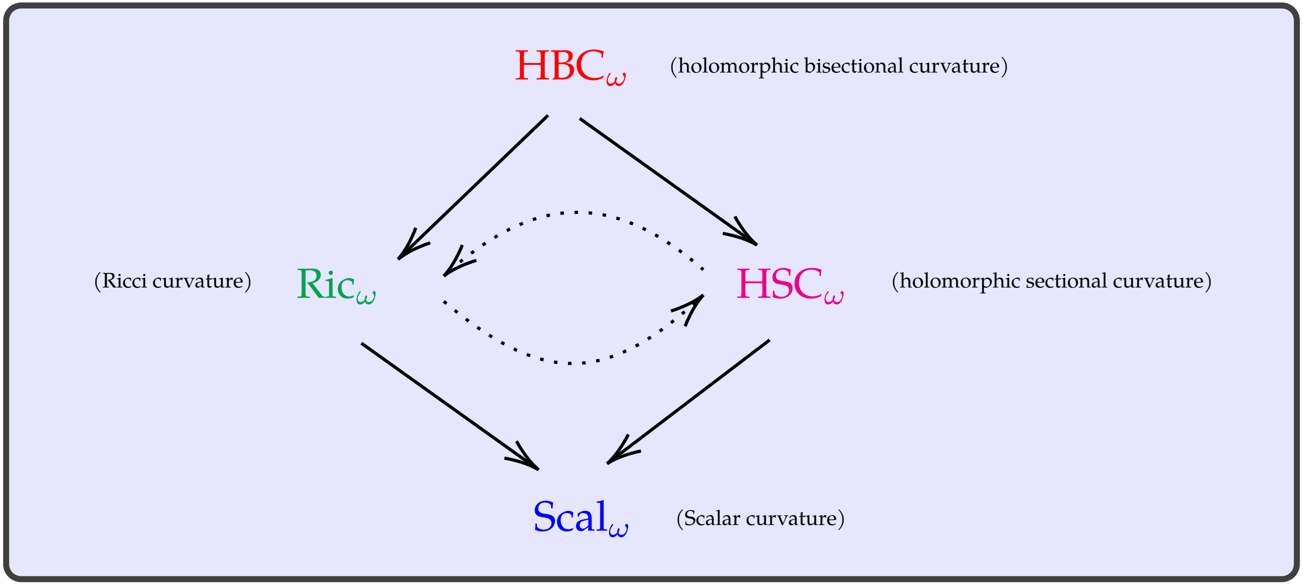
Conjecture. Let X be a *compact Kobayashi hyperbolic manifold*. Then *K_X is ample*.

Remarks:

- (Demailly 1997) Kobayashi hyperbolicity[6] is *strictly weaker* than the *existence of a metric with negative holomorphic sectional curvature*.
- Kobayashi (1970) conjectured that a *compact Kähler manifold which is Kobayashi hyperbolic has ample canonical bundle*.

[6] That is, every entire holomorphic curve $\mathbb{C} \to X$ is constant.

The *holomorphic sectional curvature* and *Ricci curvature* occupy similar strata of the curvature heirarchy[7]:



[7] Arrows indicate dominance: i.e., $A \to B$ means that $A > 0 \implies B > 0$, and similarly for < 0, ≤ 0, ≥ 0, etc.
Recall: $\mathrm{HBC}_\omega(u, v) = R(u, \overline{u}, v, \overline{v})$; $\quad \mathrm{HSC}_\omega(u) = R(u, \overline{u}, u, \overline{u})$;

Example 1. (Hitchin). Let $\mathcal{F}_n := \mathbb{P}(1 \oplus H^n)$ denote the *nth Hirzebruch surface* (a $\mathbb{P}^1$–bundle over $\mathbb{P}^1$).

Hitchin showed that $\mathcal{F}_n$ admits a Kähler metric ω with $\text{HSC}_\omega > 0$. For $n > 1$, however, $c_1(\mathcal{F}_n) \not> 0$, and thus, *does not support a Kähler metric of positive Ricci curvature*.

Example 2. Let

$$X_d := \{z_0^d + \cdots + z_n^d = 0\} \subseteq \mathbb{P}^n$$

denote the degree d *Fermat hypersurface*.

For $d \geq n+2$, adjunction implies that K_{X_d} is ample, and thus X_d *admits a Kähler(–Einstein) metric of negative Ricci curvature*. But X_d *admits complex lines*, and thus, cannot support a metric with $\text{HSC}_\omega < 0$.

To extend *Royden's* argument beyond the Kähler setting, we need to understand

$$\Delta_\omega |\partial f|^2 = |\nabla \partial f|^2 + g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta}.$$

Remarks:

- The *Monge–Ampère equation* controls the *first Chern–Ricci*

$$^c\mathrm{Ric}^{(1)}_\omega = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}.$$

- But the *second Chern–Ricci curvature* appears in the Schwarz lemma

$$^c\mathrm{Ric}^{(2)}_\omega = g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}.$$

- *Royden's polarization argument* requires the curvature of the target metric to have the symmetry

$$R_{i\bar{j}k\bar{\ell}} = R_{k\bar{j}i\bar{\ell}}.$$

  In particular, a non-Kähler metric will *not support this symmetry* in general.

Target curvature term

To understand the target curvature term

$$R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta} g^{p\overline{q}} f_p^\gamma \overline{f_q^\delta},$$

choose a frame such that at a point $p \in X$, $g_{i\bar{j}}(p) = \delta_{ij}$ and $f_i^\alpha = \lambda_i \delta_i^\alpha$, where $\lambda_i \in \mathbb{R}$. Then

$$R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta} g^{p\overline{q}} f_p^\gamma \overline{f_q^\delta} = \sum_{\alpha,\gamma} R^h_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \lambda_\alpha^2 \lambda_\gamma^2.$$

This motivated Yang–Zheng to introduce the following:

Definition. Let (X, ω) be a Hermitian manifold. The *real bisectional curvature* RBC_ω is the function

$$\mathrm{RBC}_\omega(v) := \frac{1}{|v|^2} \sum_{\alpha,\gamma} R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} v_\alpha v_\gamma,$$

where $v = (v_1, ..., v_n) \in \mathbb{R}^n \backslash \{0\}$.

The Real Bisectional Curvature

(i) If the metric is *Kähler*:
the *real bisectional curvature* is *comparable* to the *holomorphic sectional curvature*.

(ii) For a general Hermitian metric:
the *real bisectional curvature* *strictly dominates* the *holomorphic sectional curvature* and *scalar curvatures*.

(iii) The *real bisectional curvature* is *not strong enough*, however, to control the *Ricci curvatures*.

Yang–Zheng (2017) proved the following:

Theorem. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between *Hermitian manifolds*. Suppose $\text{Ric}_{\omega_g}^{(2)} \geq -C_1\omega_g + C_2 f^*\omega_h$ for constants C_1, C_2, where $C_2 \geq 0$. If $\text{RBC}_{\omega_h} \leq -\kappa_0 \leq 0$, then

$$\Delta_{\omega_g} \log |\partial f|^2 \ \geq \ -C_1 + \frac{1}{r}(C_2 + \kappa_0)|\partial f|^2.$$

Hence, if X is compact,

$$|\partial f|^2 \leq \frac{C_1 r}{(C_2 + \kappa_0)}.$$

The previous argument for the Wu–Yau theorem can be applied to show that

Corollary. (Yang–Zheng). Let X be a *compact Kähler manifold* with a *Hermitian metric* of *negative real bisectional curvature*. Then *K_X is ample*.