A General Schwarz Lemma for Hermitian Manifolds

With Applications to a Conjecture of Kobayashi and Lang

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The University of Queensland
Generalized Ricci flow Learning Seminar

The results of this talk are based on joint work with James Stanfield and some work in preparation with Frédéric Campana.

The Unit Disk $\mathbf{D}$

The unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ in the complex plane has a number of remarkable properties:

(1) Every holomorphic map $\mathbf{C} \to \mathbf{D}$ is constant.

(2) There is a complete metric on $\mathbf{D}$ with curvature bounded above by a negative constant.

(3) There is a distance function $\mathfrak{K}_{\mathbf{D}}$ for which the automorphisms of $\mathbf{D}$ are isometries and holomorphic self-maps $f : \mathbf{D} \to \mathbf{D}$ are decreasing in the sense that $f^*\mathfrak{K}_{\mathbf{D}} \leq \mathfrak{K}_{\mathbf{D}}$.

(4) The canonical bundle $K_S = \Lambda_S^{1,0}$ of a curve S universally covered by $\mathbf{D}$ is ample.

Statement (1) is just the classical Liouville theorem from one complex variable. The metric in statement (2) is the Poincaré metric of constant Gauss curvature -4. Statement (3) is a consequence of the Schwarz–Pick lemma, where $\mathfrak{K}_{\mathbf{D}}$ is the distance function obtained from integrating the Poincaré metric. Statement (4) is a consequence of the uniformization theorem and Riemann–Roch.

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The Ahlfors–Schwarz lemma argument does not show that ($\hat{2}$) $\implies$ ($\hat{3}$). Greene–Wu (1979) showed that ($\hat{2}$) $\implies$ ($\hat{3}$) by estimating $\mathfrak{K}_X$ directly. Recall that the canonical bundle K_X of a compact complex manifold X ample if the sections of $K_X^{\otimes \ell}$ furnish an embedding $\Phi : X \to \mathbf{P}^{N_\ell}$. The manifolds in ($\hat{4}$) are all projective algebraic. It is clear that condition ($\hat{4}$) is the weakest. The standard example to bring to mind is the Fermat hypersurface

$$F_d := \{z_0^d + \cdots + z_n^d = 0\} \subset \mathbf{P}^n$$

of degree $d > n+1$.

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$$F_d := \{z_0^d + \cdots + z_n^d = 0\} \subset \mathbf{P}^n$$

of degree $d > n + 1$.

Each of these properties has been used to define classes of 'hyperbolic' complex manifolds. A complex manifold X is said to be

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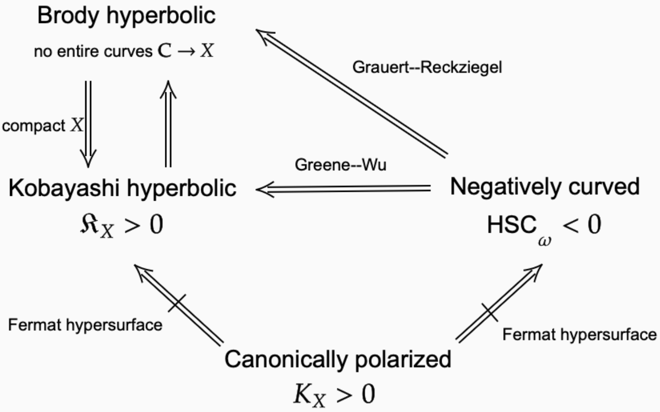
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no entire curves $\mathbb{C} \to X$
compact X
Grauert--Reckziegel
Greene--Wu
Kobayashi hyperbolic
$\mathfrak{K}_X > 0$
Negatively curved
$\text{HSC}_\omega < 0$
Fermat hypersurface
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Canonically polarized
$K_X > 0$

The main purpose of this talk is to present the most general evidence for the following conjecture of Kobayashi (1970) and Lang (1986).

Conjecture. Let X be a compact Kobayashi hyperbolic Kähler manifold. Then the canonical bundle K_X is ample.

Recall that a complex manifold X is Kähler if it admits a Hermitian metric g such that the 2-form $\omega(\cdot, \cdot) := g(J\cdot, \cdot)$ is closed.

A positive resolution of the Kobayashi–Lang conjecture has a number of profound implications:

(i) All compact Kobayashi hyperbolic Kähler manifolds are projective.

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Examples of Kobayashi hyperbolic manifolds

We will be exclusively interested in compact Kobayashi hyperbolic manifolds. Hence, a compact complex manifold X is Kobayashi hyperbolic if every holomorphic map $\mathbf{C} \to X$ is constant.

Examples:

- Compact quotients of bounded domains $\Omega \subset \mathbf{C}^n$.
- A generic smooth hypersurface of degree $d \geq 16n^3(5n+4)$ in $\mathbf{P}^{n+1}$.
  - This lower bound is due to Bérczi–Kirwan (2023); building on the work of Siu (2015), Brotbek (2017), Deng (2017), Demailly (2018), and others.
- Kobayashi hyperbolicity is inherited by products, submanifolds, universal covers, and fiber spaces (i.e., if the base and fibers are hyperbolic, then the total space is hyperbolic).
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The techniques to address the Kobayashi–Lang conjecture (and its folklore generalization) vastly diverge depending on the additional assumptions one places on the manifold.

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(2) Surfaces, where the focus is on producing rational curves on class VII_0 surfaces with $b_2 \geq 3$.

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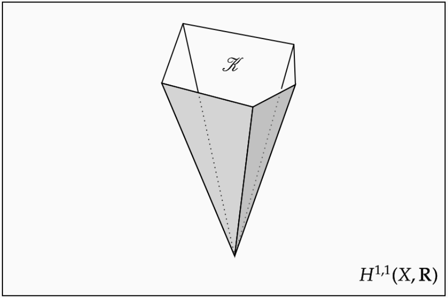
The closure of the Kähler cone $\overline{\mathscr{K}}$ defines the nef cone.



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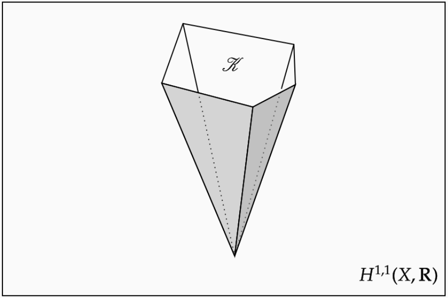

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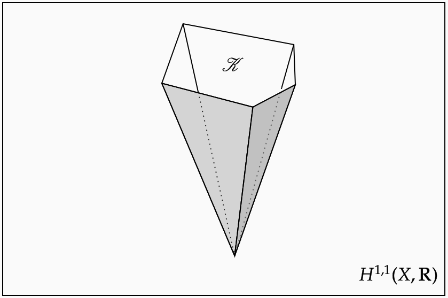

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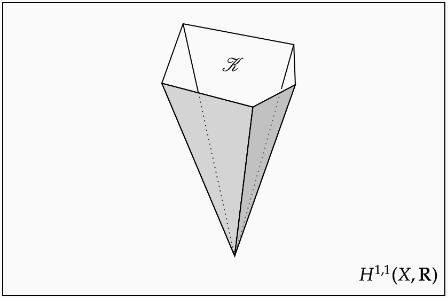

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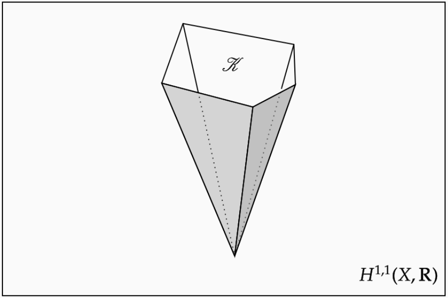

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The (Kähler extension of the) abundance conjecture predicts that K_X being nef is equivalent to K_X being semi-ample, i.e., the sections of $K_X^{\otimes \ell}$, for some $\ell \in \mathbf{N}$, define a surjective holomorphic map $\Phi : X \to Y \subset \mathbf{P}^{N_\ell}$ with connected fibers (a fiber space). This is known for projective manifolds of dimension ≤ 3 (Kawamata, Kollár, Miyaoka, and others).

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If X is a compact Kobayashi hyperbolic manifold with $\kappa_X = \dim_{\mathbf{C}} X$, the Iitaka map $\Phi : X \to Y$ is a bimeromorphic map. Hence, X is Moishezon (bimeromorphic to a projective variety), and since X has no rational curves, it is projective (BCHM 2010, Cascini 2013). From the base-point-free theorem and the relative cone theorem (Mori 1982, Kawamata 1985, Kollár–Miyaoka–Mori 1992), K_X is ample. Hence, compact Kobayashi hyperbolic manifolds with $\kappa_X = \dim_{\mathbf{C}} X$ have ample canonical bundle (X. Yang 2020).

If X is projective Kobayashi hyperbolic with $\kappa_X < \dim_{\mathbf{C}} X$, it suffices to rule out the hyperbolicity of the smooth fibers of the Iitaka map, which are projective manifolds with holomorphically torsion canonical bundle.

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The universal cover of a compact Kähler manifold with holomorphically torsion canonical bundle splits as a product of Euclidean, Calabi–Yau, and hyperkähler factors (Bogomolov 1974, Beauville 1983).

- A simply connected projective manifold X is said to be Calabi–Yau if $K_X \simeq \mathcal{O}_X$ and $H^0(X, \Omega_X^\bullet) = \mathbf{C} \oplus \mathbf{C}\alpha$, where α is a generator of K_X.
- A simply connected compact complex manifold is hyperkähler if $H^0(X, \Omega_X^\bullet) = \mathbf{C}[\sigma]$, where $\sigma \in H^0(X, \Omega_X^2)$ is everywhere non-degenerate.

Verbitsky (2015) showed that hyperkähler manifolds with $b_2 > 3$ are not hyperbolic. It is conjectured that all hyperkähler manifolds have $b_2 > 3$.

The non-hyperbolicity of K3 surfaces was shown by Wong (1981) and Campana (1991).

If a Kobayashi hyperbolic Calabi–Yau threefold exists, it would be forced to satisfy: $b_2 \leq 13$, together with several constraints on its second Chern class $c_2(X)$ and its Kähler cone $\mathscr{K}$ (Wilson 1989, Peternell 1991, Heath-Brown–Wilson 1992, Diverio–Ferretti 2012, and others). If the Kawamata–Morrison cone conjecture holds, then $b_2 = 1$ (Diverio 2013).

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Non-Kähler Surfaces

While the Kobayashi–Lang conjecture is only asserted in the category of compact Kähler manifolds, it is a matter of folklore that the conjecture should hold for all compact Kobayashi hyperbolic manifolds.

It is true for non-Kähler compact complex surfaces with $b_2 < 3$ (Bogomolov 1976, Li–Yau–Zheng 1994, Teleman 1994, 2005, 2009). Further, if the global spherical shell conjecture holds, then this would imply the folklore conjecture in dimension two (Dloussky–Oeljeklaus–Toma 2003).

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Curvature Aspects of Hyperbolicity

Let X be a complex manifold. Let g be a Hermitian metric, locally described in a coordinate chart $(z_1, ..., z_n)$ by

$$\mathrm{g} = \sum_{k,\ell} \mathrm{g}_{k\bar{\ell}} dz^k \otimes d\bar{z}^\ell,$$

where $\mathrm{g}_{k\bar{\ell}} = \mathrm{g}\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_\ell}\right)$ is a Hermitian matrix.

The Chern curvature tensor of g is the $(0,4)$–tensor whose components are locally given by

$$\mathrm{R}_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 \mathrm{g}_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_j} + \mathrm{g}^{p\bar{q}} \frac{\partial \mathrm{g}_{k\bar{q}}}{\partial z_i} \frac{\partial \mathrm{g}_{p\bar{\ell}}}{\partial \bar{z}_j}.$$

The holomorphic sectional curvature of a Hermitian metric g is defined

$$\mathrm{HSC}_{\mathrm{g}}(\xi) = \frac{1}{|v|_{\mathrm{g}}^4} \sum_{i,j,k,\ell} \mathrm{R}_{i\bar{j}k\bar{\ell}} \xi^i \overline{\xi}^j \xi^k \overline{\xi}^\ell,$$

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Examples of compact complex manifolds with HSC < 0:

- The Bergman metric on compact quotients of a bounded symmetric domain.
- Inherited by submanifolds, products, and submersions, i.e., if the base and fiber of a holomorphic submersion have HSC< 0, then so does the total space (Cheung 1988).
- (Mohsen 2022). Let X be a projective manifold with $\dim_{\mathbf{C}} X = n$. If $n \geq 3d$, then for every sufficiently large k, there is a complete intersection Y of dimension d defined by equations of degree k which has HSC < 0.

The holomorphic sectional curvature (of a Kähler metric) dominates the scalar curvature. But, in general, it only dominates the sum of the scalar curvatures $\mathrm{Scal}_g + \widetilde{\mathrm{Scal}}_g$, where $\mathrm{Scal}_g := g^{i\bar{j}} g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}$ and $\widetilde{\mathrm{Scal}}_g := g^{i\bar{\ell}} g^{k\bar{j}} R_{i\bar{j}k\bar{\ell}}$.

Hirzebruch surfaces $\mathcal{F}_n := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$ for $n > 1$ have Kähler metrics with HSC > 0 but no Kähler metrics of positive Ricci curvature (Hitchin 1975).

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The Wu–Yau Strategy

The Wu–Yau strategy makes use of the Kähler cone $\mathscr{K}$ to produce a sequence of metrics with Ricci curvature bounded from below.

Proceed by contradiction and suppose that K_X is not nef, i.e., $c_1(K_X) \notin \overline{\mathscr{K}}$. Then for some $\varepsilon_0 > 0$, the cohomology class $\varepsilon_0\{\omega_0\} + c_1(K_X)$ lies on the boundary of the nef cone $\overline{\mathscr{K}}$.

Hence, for any $\varepsilon > 0$, we have a Kähler class $(\varepsilon + \varepsilon_0)\{\omega_0\} + c_1(K_X)$. By the Aubin–Yau theorem, we have a sequence of Kähler metrics

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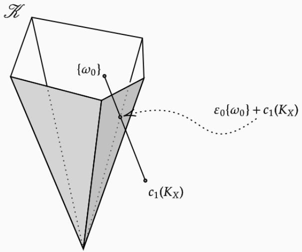

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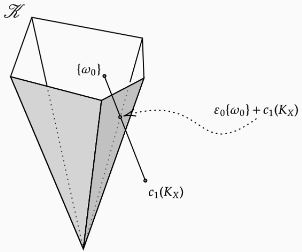

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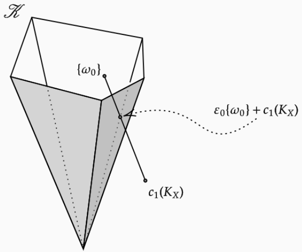

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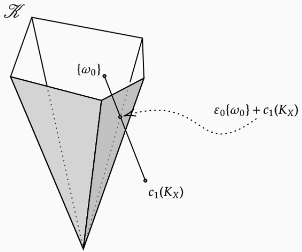

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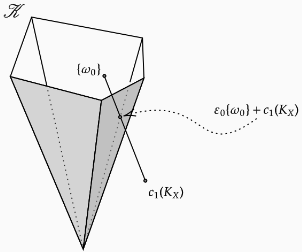

Differentiating the Monge–Ampère equation $\omega_\varepsilon^n = e^{u_\varepsilon}\omega_0^n$ implies that

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The crux of the argument is the Schwarz lemma, i.e., an estimate on $\mathrm{tr}_{\omega_\varepsilon}(f^*\widehat{\omega})$, where $f = \mathrm{id} : (X, \omega_\varepsilon) \to (X, \widehat{\omega})$ is the identity map and $\widehat{\omega}$ is the Kähler metric of negative holomorphic sectional curvature.

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Lu (1967) showed that

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Two general improvements

Since Lu's calculation in 67, there have been two general improvements:

- Yau (1978) applied his maximum principle to this calculation, which permitted significantly more general source manifolds.
- Royden (1980) showed that the target curvature term is controlled from an upper bound on the holomorphic sectional curvature if the target metric is Kähler.[1] This permits significantly more general target manifolds.

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Royden's Schwarz Lemma

Theorem. (Royden). Let X be a compact complex manifold with $\dim_{\mathbf{C}} X = n$. Let $f : (X, g) \to (X, \widehat{g})$ be a holomorphic map. Suppose that g is Kähler with

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and hence,

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Extracting a smooth limit as $\varepsilon \searrow 0$ yields the desired contradiction, showing that K_X is nef. The same argument, assuming that K_X is nef but not ample, establishes the ampleness of the canonical bundle if the holomorphic sectional curvature is strictly negative.

The Kähler assumption is used in a number of ways:

- (†) To set-up the Monge–Ampère equation (from the Kähler cone).
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The Real Bisectional Curvature

In the Wu–Yau theorem, the negatively curved metric is only used to control the target curvature term

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in Royden's Schwarz lemma For non-Kähler metrics, Royden's argument fails to control this term. Yang–Zheng (2018) introduced the real bisectional curvature

$$\mathrm{RBC}_{\widehat{\mathrm{g}}}(\xi) \ := \ \frac{1}{|\xi|^2} \sum_{\alpha,\beta,\gamma,\delta} \widehat{\mathrm{R}}_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha\bar{\beta}} \xi^{\gamma\bar{\delta}},$$

precisely to control this target curvature term. As a consequence, Yang–Zheng proved the following extension of the Wu–Yau theorem.

Theorem. (Yang–Zheng). Let X be a compact Kähler manifold with a Hermitian metric of $\mathrm{RBC}_{\widehat{\mathrm{g}}} < 0$. Then X has ample canonical bundle.

Because the real bisectional curvature is defined to be the curvature constraint that appears as the target curvature term in the Schwarz lemma, it is far from clear if any improvements can be made.

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Theorem. (Yang–Zheng). Let X be a compact Kähler manifold with a Hermitian metric of $\mathrm{RBC}_{\widehat{\mathrm{g}}} < 0$. Then X has ample canonical bundle.

Because the real bisectional curvature is defined to be the curvature constraint that appears as the target curvature term in the Schwarz lemma, it is far from clear if any improvements can be made.

The Real Bisectional Curvature

In the Wu–Yau theorem, the negatively curved metric is only used to control the target curvature term

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But it turns out that the real bisectional curvature is not sharp, and the purpose of the present talk is to exhibit the first general improvement on the Schwarz lemma in the Hermitian category since Royden.

Before stating the main workhorse, let us state the main application; the following most general form of the Kobayashi–Lang conjecture:

Theorem. (B.–Stanfield, 2023). Let X be a compact Kähler manifold with a pluriclosed metric $\widehat{g}$ of $\text{HSC}_{\widehat{g}} < 0$. Then X has ample canonical bundle.

Recall that a Hermitian metric ω is pluriclosed if $\partial\bar{\partial}\omega = 0$. Such metrics always exist on a compact complex surface (Gauduchon). The pluriclosed condition is the only non-Kähler condition I'm aware of that is preserved by the map $\omega_0 \mapsto (\varepsilon + \varepsilon_0)\omega_0 - \text{Ric}^{(1)}_{\omega_0} + \sqrt{-1}\partial\bar{\partial}u_\varepsilon$

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The general Schwarz lemma is the following:

Theorem. (B.–Stanfield, 2023). Let $f : (X, g) \to (Y, \widehat{g})$ be a holomorphic map between Hermitian manifolds. For $\tau > 0$, we have

$$\begin{aligned} \Delta_g \mathrm{tr}_g(f^* \widehat{g}) \quad &\geq \quad \left(\mathrm{Ric}^{(2)}_{k\bar{\ell}} + \frac{1}{4}(1 - 1/\tau) \mathcal{Q}^2_{k\bar{\ell}} \right) g^{k\bar{q}} g^{p\bar{\ell}} f^\alpha_p \overline{f^\beta_q} \widehat{g}_{\alpha\bar{\beta}} \\ &\quad - \left(\widehat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{1}{4}(1-\tau) \widehat{T}^\rho_{\alpha\gamma} \overline{\widehat{T}^\sigma_{\beta\delta}} \widehat{g}_{\rho\bar{\sigma}} \right) g^{i\bar{j}} f^\alpha_i \overline{f^\beta_j} g^{p\bar{q}} f^\gamma_p \overline{f^\gamma_q}, \end{aligned}$$

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If the source metric g is Kähler, then we may take $\tau = 0$. The new target curvature term is then what we call the tempered real bisectional curvature

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is intrinsic to the Hermitian structure in the sense that it is the second-order term for the Hessian of the metric in geodesic normal coordinates for the Chern connection.

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Hence, we have the following Schwarz lemma:

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One of the most significant achievements in understanding (a priori) non-Kähler Kobayashi hyperbolic manifolds was achieved by Lee–Streets (2021). Using the pluriclosed flow, they were able to establish the following remarkable result.

Theorem. (Lee–Streets). Let X be a compact pluriclosed manifold with a Hermitian metric $\widehat{g}$ of $\text{RBC}_{\widehat{g}} < 0$. Then X has ample canonical bundle.

There is a parabolic extension of the tempered Schwarz lemma:

Theorem. (B.–Stanfield). Let X be a complex manifold with a smooth family of Hermitian metrics g_t satisfying

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for some $\tau > 0$ sufficiently small. Let $\widehat{g}$ be a pluriclosed metric with $\text{HSC}_{\widehat{g}} \leq -\Lambda_0 < 0$. Then

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Compatibility with the pluriclosed flow

Let g_t be a family of Hermitian metrics evolving under the pluriclosed flow

$$\frac{\partial g_t}{\partial t} = -\mathrm{Ric}^{(2)}_{g_t} + \mathcal{Q}^{(1)}_{g_t},$$

where $\mathcal{Q}^{(1)}_{k\bar{\ell}} = T^r_{kp}\overline{T^s_{\ell q}} g_{r\bar{s}} g^{p\bar{q}}$. The parabolic tempered Schwarz lemma applied to the pluriclosed flow yields

$$(\partial_t - \Delta_{g_t})\,\mathrm{tr}_{g_t}(\widehat{g}) \leq -\mathcal{Q}^{(1)} - \underbrace{\frac{1}{4}(1-1/\tau)\mathcal{Q}^2}_{\text{bad term}} + \mathrm{tr}_{g_t}(\widehat{g}) + \mathrm{RBC}^{\tau}_{\widehat{g}}.$$

Note that we require $\tau > 0$ to be small to ensure that $\mathrm{HSC}_{\widehat{g}} < 0 \implies \mathrm{RBC}_{\widehat{g}} < 0$ if $\widehat{g}$ is pluriclosed. If this technicality can be handled, it would follow that a compact complex manifold with a pluriclosed metric of negative holomorphic sectional curvature has ample canonical bundle.

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