A Locality Theorem for Einstein Metrics on Compact Complex Manifolds

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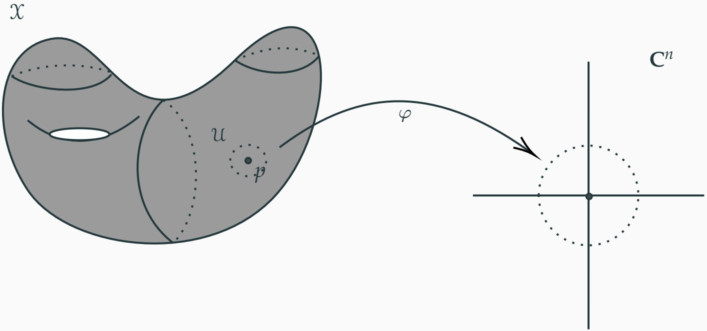
The aim of this talk is to report on some recent joint work with Artem Pulemotov (The University of Queensland).

A complex manifold $\mathcal{X}$ is a space that is locally modelled on $\mathbf{C}^n$ in such a way that the local analytic structure is preserved.



Examples: Euclidean space $\mathbf{C}^n$; projective space $\mathbf{P}^n$; the ball $\mathbf{B}^n$.

Let $\mathcal{X}$ be a compact complex manifold with an Einstein metric

$$\mathrm{Ric}(\omega) \; = \; \lambda\omega, \qquad\qquad \lambda \in \mathbf{R}.$$

Suppose that $\mathcal{X}$ admits another metric $\widetilde{\omega}$ with some 'good' property.

The Locality Problem. When does the Einstein metric ω inherit this good property from the mere existence of $\widetilde{\omega}$?

Riemannian Metrics

Definition. A Riemannian metric g on $\mathcal{X}$ is smooth assignment of positive-definite quadratic forms g_p on each tangent space $T_p\mathcal{X}$.

The Riemannian metric allows us to compute the lengths of tangent vectors, and by integrating, the lengths of curves in the manifold. This provides a notion of distance.

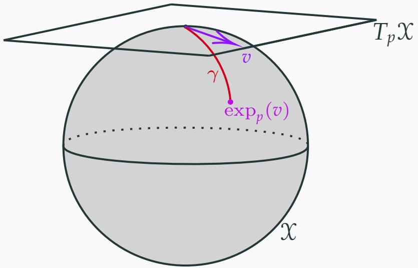
A geodesic is a curve in $\mathcal{X}$ which locally minimizes the distance between any two points.

The Exponential Map

The exponential map

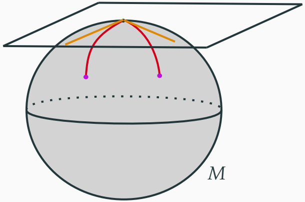
$$\exp_p : T_p\mathcal{X} \to \mathcal{X}, \qquad \exp_p(v) := \gamma(1),$$

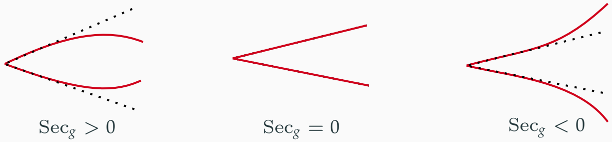
where $\gamma : [0,1] \to \mathcal{X}$ is unique geodesic satisfying $\gamma(0) = p$ and $\gamma'(0) = v$.



The exponential map provides a canonical set of coordinates on a Riemannian manifold.

The sectional curvature measures the failure of the exponential map to be an isometry.

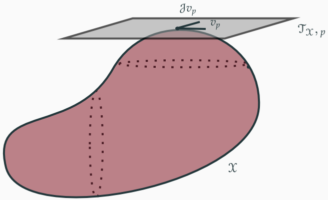


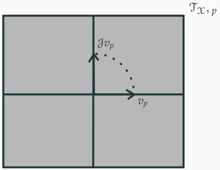


The Ricci curvature measures the extent to which the exponential map distorts volumes.

Complex Structures

The complex structure of a complex manifold $\mathcal{X}$ can be encoded in an endomorphism $\mathcal{J} \in \mathrm{End}(\mathcal{T}_{\mathcal{X}})$ satisfying $\mathcal{J}^2 = -\mathrm{id}$ together with an integrability criterion.





Definition. A Riemannian metric g is said to be Hermitian if

$$g(\mathcal{J}\cdot, \mathcal{J}\cdot) \;=\; g(\cdot, \cdot).$$

If the 2-form $\omega(\cdot, \cdot) := g(\mathcal{J}\cdot, \cdot)$ is symplectic, i.e., $d\omega = 0$, then g is said to be Kähler.

Kähler Examples: $\mathbf{C}^n$; $\mathbf{B}^n$; $\mathbf{P}^n$; submanifolds (hence, projective and Stein manifolds are Kähler).

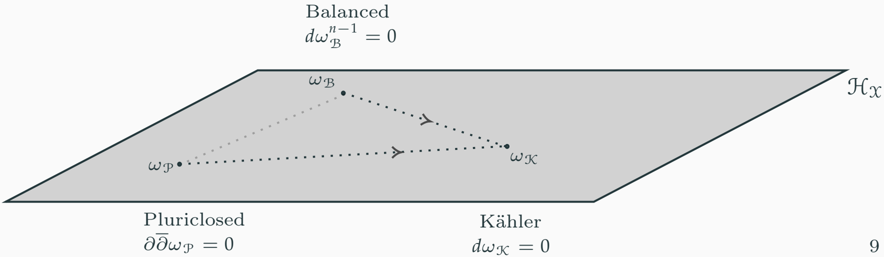
Non-Kähler Examples: The Hopf manifolds $\mathbf{S}^1 \times \mathbf{S}^{2n-1}$, for $n > 1$.

Definition. A Hermitian metric ω is said to be

- pluriclosed if $\partial\bar{\partial}\omega = 0$.
- balanced if $d\omega^{n-1} = 0$.

A metric that is simultaneously pluriclosed and balanced is Kähler.

Conjecture. (Fino–Vezzoni). A compact complex manifold with a pluriclosed metric $\omega_{\mathcal{P}}$ and a balanced metric $\omega_{\mathcal{B}}$ admits a Kähler metric $\omega_{\mathcal{K}}$.



Kähler–Einstein Metrics

Theorem. (Aubin–Yau, 76). Let $\mathcal{X}$ be a compact Kähler manifold with $c_1(-K_{\mathcal{X}}) < 0$. Then there exists a Kähler–Einstein metric ω_{KE} with

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}.$$

In particular, compact Kähler manifolds with $\mathrm{Sec}(\omega) < 0$ admit Kähler–Einstein metrics.

Theorem. (Yau, 76). Let $\mathcal{X}$ be a compact Kähler manifold with $c_1(-K_{\mathcal{X}}) = 0$. Then there exists a Kähler–Einstein metric ω_{KE} with

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = 0.$$

Compact Kähler manifolds with $c_1(-K_{\mathcal{X}}) = 0$ are said to be Calabi–Yau.

The Wu–Yau Theorem

In complex geometry, we may restrict the sectional curvature to the 2-planes $\{u, \mathcal{J}u\}$ invariant under the complex structure.

This significantly weaker constraint defines the holomorphic sectional curvature

$$\mathrm{HSC}_\omega := \frac{1}{|u|_\omega^4} R(u, \mathcal{J}u, u, \mathcal{J}u).$$

<u>Theorem</u>. (Wu–Yau, 16). Let $(\mathcal{X}, \omega)$ be a compact Kähler manifold with $\mathrm{HSC}(\omega) < 0$. Then there is a Kähler–Einstein metric ω_{KE} with

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}.$$

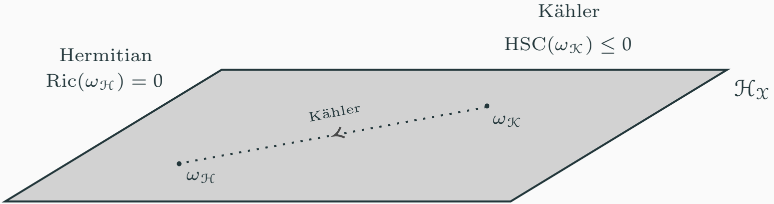
The locality problem is difficult because there is no assumed relationship between the metrics.

The aim of the present talk is to give a new (and more robust) technique to address locality problems.

Application 1

The first application of this new technique:[1]

Theorem A. Let $(\mathcal{X}, \omega_{\mathcal{K}})$ be a compact Kähler manifold with $\mathrm{HSC}_{\omega_{\mathcal{K}}} \leq 0$. If $\mathcal{X}$ admits a Ricci-flat Hermitian metric $\omega_{\mathcal{H}}$ with $\mathrm{Ric}(\omega_{\mathcal{H}}) = 0$, then $\omega_{\mathcal{H}}$ is Kähler–Einstein and $\mathcal{X}$ is Calabi–Yau.



[1] For the experts: For a Hermitian metric ω, we understand $\mathrm{Ric}(\omega)_{k\bar{\ell}} = g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}$ to be the second Chern Ricci curvature.

Application 2

The second application verifies the Fino–Vezzoni conjecture under curvature conditions. [2]

Theorem B. If $(\mathcal{X}, \omega_{\mathcal{B}})$ is a compact Ricci-flat balanced manifold with $\mathrm{Ric}(\omega_{\mathcal{B}}) = 0$. If $\mathcal{X}$ admits a pluriclosed metric $\omega_{\mathcal{P}}$ of $\mathrm{RBC}(\omega_{\mathcal{P}}) \leq 0$. Then $\omega_{\mathcal{P}}$ is a Kähler metric.

[2] Recall that the Fino–Vezzoni conjecture predicts that if a compact complex manifold admits both a balanced ($d\omega^{n-1} = 0$) and pluriclosed metric ($\partial\bar{\partial}\omega = 0$) metric, it admits a Kähler metric ($d\omega = 0$).

1. Many locality problems can be addressed by showing that two metrics have the same Chern connection.

If a metric has the same Chern connection[3] as a Kähler metric, the metric is also Kähler.

2. Determining whether two metrics have the same connection can be addressed via the Bochner technique.

This was observed in the Riemannian category by DeTurck–Koiso (84).

[3] The Chern connection is the unique connection ∇ such that $\nabla g = \nabla J = 0$ and whose torsion satisfies $T(J\cdot, \cdot) = JT(\cdot, \cdot)$.

Further Applications: Non-standard Einstein Metrics on $\mathbf{P}^n$

Problem: Does $\mathbf{P}^n$ admit an Einstein metric g with $\mathrm{Ric}(g) = -g$?

Known results:

- Lohkamp (94): For $n > 1$, there is a Riemannian metric with $\mathrm{Ric} < 0$ on $\mathbf{P}^n$.
- Bando–Mabuchi (86): Kähler–Einstein metric on $\mathbf{P}^n$ are biholomorphically isometric to the standard Fubini–Study metric.
- LeBrun (2021) showed that Hermitian Einstein metrics on $\mathbf{P}^2$ are Kähler–Einstein.

Extending our methods, we expect to be able to show that there are no Einstein metrics on $\mathbf{P}^n$ with negative Ricci curvature.