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Laplacian Estimates for Holomorphic Maps between Hermitian Manifolds

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Abstract

We survey the recent developments around the Schwarz lemma for holomorphic maps. The Schwarz lemma provides a tool for restricting the existence of holomorphic maps between complex manifolds. We begin with a preliminary discussion regarding complex manifolds, holomorphic vector bundles, Hermitian metrics, and curvature of Hermitian metrics. This culminates in a discussion of the Schwarz lemmas. We first consider those of Chern–Lu type, specifically those of Royden, Yang–Zheng, and Broder–Stanfield Schwarz lemmas. The Aubin–Yau Schwarz lemmas are then discussed, with the Broder Schwarz lemma being the primary focus. Several corollaries of the Schwarz lemmas are provided throughout.

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Contents

A complex manifold is a topological space locally modelled on $\mathbb{C}^n$ by maps which are *holomorphic* (i.e., preserve the complex-analytic structure). Studying the maps into and out of complex manifolds is central to understanding all such spaces. In particular, we restrict our focus to *holomorphic functions* and *holomorphic curves* i.e., non-constant holomorphic maps $X \to \mathbb{C}$ and $\mathbb{C} \to X$ respectively. The presence or absence of these maps naturally organises complex manifolds into four classes:

(i) holomorphic functions $X \to \mathbb{C}$ are abundant;
(ii) there are no holomorphic curves $\mathbb{C} \to X$;
(iii) holomorphic curves $\mathbb{C} \to X$ are abundant;
(iv) there are no holomorphic functions $X \to \mathbb{C}$.

Class (iv) is far too large to admit any insightful study; indeed, all compact complex manifolds lie in (iv) by the maximum principle. This observation reduces the paradigm to the study of three classes. Class (i) is occupied by complex manifolds which admit holomorphic embeddings into $\mathbb{C}^N$ for some $N \in \mathbb{N}$, referred to as *Stein manifolds*. Forstnerič's *Oka manifolds* [For09] and Campana's *special manifolds* [Cam04] arise in class (iii). Class (ii) is significant enough to warrant a definition of its own - *Brody hyperbolic manifolds* [Bro78]. For this project, we narrow our focus to manifolds of this type. Hence, restricting holomorphic maps between complex manifolds becomes our primary goal. The Schwarz lemma provides the key tool for this restriction.

The principle of restricting holomorphic maps has its origins in complex analysis. In particular, this principle is exemplified by the classical Schwarz lemma, a pioneering result in the field. It asserts that for every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(0) = 0$, where $\mathbb{D}$ is the unit disk, two main conclusions are reached:

(1) $|f(z)| \leq |z|$ for all z in the unit disk;
(2) If the equality holds for any $z \neq 0$, then f is a rotation of the unit disk about the origin.

The classical Schwarz lemma imposes strong restrictions on the behaviour of holomorphic self-maps of the unit disk. This result was extended in [Pic16] with the *Schwarz–Pick lemma*, when it was observed that f need not fix the origin. This resulted in a different type of constraint: holomorphic self-maps of the disk are distance decreasing under the *Poincaré distance* ρ:

$$d_\rho(f(z), f(w)) \leq d_\rho(z, w).$$

The first incarnation of a Schwarz lemma in differential geometry was considered by Ahlfors [Ahl38], generalising the results of Pick. We now consider holomorphic maps $f : \mathbb{D} \to (\Sigma, g)$, where (Σ, g) is a Riemann surface with a Hermitian metric g. In such a case, the familiar Gauss curvature κ_g is our primary tool for restricting holomorphic maps. If $\kappa_g \leq -1$, we obtain the *Schwarz–Ahlfors–Pick lemma*, or simply the *Ahlfors–Schwarz lemma*:

$$d_g(f(z), f(w)) \leq d_\rho(z, w).$$

That is, all holomorphic maps into a negatively curved Riemann surface are distance decreasing with respect to the Poincaré distance. One may interpret the Ahlfors–Schwarz lemma as an extension of the Schwarz–Pick lemma to a higher dimensional setting; imbuing the disk with the Poincaré metric given by

$$\rho = \frac{|dz|}{(1-|z|^2)},$$

the original Schwarz–Pick lemma is obtained. The Schwarz lemma posed by Ahflors marks an important theme: curvature acts as the fundamental constraint on holomorphy.

Modern Schwarz lemmas have shifted their focus away from distance decreasing maps. Instead, results regarding the non-existence of holomorphic maps and curvature properties of manifolds are considered. This is achieved through Laplacian estimates of the energy density $|\partial f|^2$. If the Laplacian is with respect to the source metric, it is referred to as *Chern–Lu* [Lu68]. Applying the Laplacian that is with respect to the target metric yields the second family of Schwarz lemmas, referred to as *Aubin–Yau* [Aub78], [Yau78].

The first of the Chern–Lu Schwarz lemmas, introduced in [Lu68], primarily focused on holomorphic maps between *Kähler manifolds*, a special class of complex manifolds where the Riemannian and complex structure interact in a compatible manner. Where it diverges from the Ahlfors' Schwarz lemma is that suitable curvature bounds are now required on target *and* source manifolds to restrict holomorphic maps. The source manifold requires a positive lower bound by the *Ricci curvature*. For the target term, however, we require a negative upper bound on the less familiar *holomorphic bisectional curvature* [GK67], defined as

$$\mathrm{HBC}_g(u,v) := \frac{1}{|u|_g^2|v|_g^2} R(u,\overline{u},v,\overline{v}) = \frac{1}{|u|_g^2|v|_g^2} \sum_{i,\overline{j},k,\overline{\ell}} R_{i\overline{j}k\overline{\ell}} u^i \overline{u}^j v^k \overline{v}^\ell$$

where u, v are tangent vectors in the holomorphic tangent bundle, and R denotes the curvature of the Chern connection. The holomorphic bisectional curvature imposes strong restrictions on the geometry of the manifold. For example, the Mori [Mor79], Siu–Yau [SY80] resolution of the Frankel conjecture demonstrates if (X, g) is a compact Kähler manifold with a Kähler metric of $\mathrm{HBC}_g > 0$, then X must be biholomorphic to n-dimensional projective space $\mathbb{P}^n$. The *holomorphic sectional curvature* HSC_g, obtained by restricting to the diagonal of HBC_g, is comparatively a much weaker curvature term. The holomorphic sectional curvature is of particular interest as it provides a large number of Brody hyperbolic manifolds, evidenced by the following corollary of Greene–Wu [GW79].

Corollary ([GW79])**.** Let (X, g) be a Hermitian manifold with $\mathrm{HSC}_g \leq -\Lambda_0 < 0$. Then X admits no entire curves i.e., X is Brody hyperbolic.

Royden [Roy80] demonstrated using purely linear-algebraic techniques[1] that HSC of the target manifold provides sufficient control for the Chern–Lu Schwarz lemma. Royden's Schwarz lemma immediately yields the following corollary:

Corollary ([Roy80])**.** Let $f : (X, g) \to (Y, h)$ denote a holomorphic map between Kähler manifolds, where X is compact. Assume that $\mathrm{Ric}_g \geq Cg$ for $C \geq 0$, and $\mathrm{HSC}_h \leq -\Lambda_0 < 0$. Then, f is constant.

Royden's trick also works when f is a holomorphic curve (or more generally, a holomorphic map of rank 1). Transitioning from a Kähler setting to a Hermitian (non-Kähler) setting for general holomorphic maps, however, poses significant problems. For a holomorphic map between Hermitian manifolds, suitable curvature bounds on the source manifold are provided by the *(second) Chern Ricci curvature*. Unlike the Kähler case, it is unknown whether the holomorphic sectional curvature

[1] This technique is usually referred to as *Royden's trick* in the literature (see, e.g., [Bro22a]).

provides suitable bounds on the target manifold. This prompted Yang–Zheng [YZ19] to study a curvature term which they coined the *real bisectional curvature*:

$$\mathrm{RBC}_g(\xi) := \frac{1}{|\xi|_g^2} \sum_{i,\bar{j},k,\bar{\ell}} R_{i\bar{j}k\bar{\ell}} \xi^{i\bar{j}} \xi^{k\bar{\ell}},$$

where ξ is a non-negative Hermitian (1,1)-tensor. A sign on the real bisectional curvature, $\mathrm{RBC}_g < 0$ for instance, forces that same sign on the holomorphic sectional curvature. If the metric is Kähler, then the converse also holds. When considering this novel curvature term, we obtain a refined version of the Chern–Lu Schwarz lemma for Hermitian metrics. This Schwarz lemma then immediately yields the following result.

Corollary ([YZ19])**.** Let $f : (X, g) \to (Y, h)$ denote a holomorphic map between Hermitian manifolds, where X is compact. Assume that $\mathrm{Ric}_g^{(2)} \geq Cg$ for $C > 0$, and $\mathrm{RBC}_h \leq -\Lambda_0 < 0$. Then, f is constant.

Further variations of the Royden Schwarz lemma (i.e., bounds regarding the holomorphic sectional curvature of the target) were considered in [BS23] for *pluriclosed manifolds* - manifolds which are not restricted by the Kähler condition, yet admit enough structure to be amenable to study. If the source is assumed to be Kähler and the target is assumed to be pluriclosed, sufficient control is provided by the HSC of the target.

The second family of Schwarz lemmas arise under the assumption that f is a biholomorphism i.e., an invertible holomorphic map with a holomorphic inverse. Further, the Laplacian of the energy density is taken with respect to the target metric. In this case, we refer to such inequalities as *Aubin–Yau*. Such estimates were originally considered by Yau [Yau78] and Aubin [Aub78] for problems involving Kähler–Einstein metrics. Several further uses have since been discovered. In the case where the target is Kähler, Yau observed that suitable curvature assumptions are provided by the holomorphic bisectional curvature. The curvature conditions within the Hermitian class are not well understood, prompting Broder [Bro22b] to consider a novel curvature term, the *Schwarz bisectional curvature*:

$$\mathrm{SBC}_g(\xi) := \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} \xi^{i\bar{j}} (\xi^{-1})^{k\bar{\ell}},$$

where ξ is a non-negative Hermitian (1,1)-tensor. Adopting this curvature term then yields a Hermitian Aubin–Yau Schwarz lemma. Applying this Schwarz lemma yields the following result.

Corollary ([BS23])**.** Let $f : (X, g) \to (Y, h)$ denote a holomorphic map between Hermitian manifolds, where X is compact and f is biholomorphic onto its image. Assume that $\mathrm{Ric}_h^{(2)} \leq -Ch$ for $C > 0$, and $\mathrm{SBC}_g \geq 0$. Then, f is constant.

With extensive developments around the Schwarz lemma in recent years, understanding and applying them becomes overwhelming to outside purveyors of the subject. Hence, we find it timely to survey the development and some notable applications of the novel Schwarz lemmas. In this thesis, we build the knowledge required to understand the contemporary Schwarz lemmas and survey the current landscape regarding the topic.

Outline of the Thesis

Chapter 1 states the preliminary material required to understand the Schwarz lemma. The first section discusses the basic theory of several complex variables and complex manifolds, where multiple examples of complex manifolds are stated. In the second section, we consider structures that accompany complex manifolds; the complexified tangent bundle, the space of (p, q)-forms, and the Dolbeault operators are studied. The section then concludes with a discussion of holomorphic vector bundles and the Koszul–Malgrange theorem. Section 3 introduces connections and curvature on holomorphic vector bundles. We construct the Chern connection on holomorphic vector bundles. We conclude this section by studying Hermitian and Kähler metrics. The final section presents the basic theory regarding curvature of Hermitian metrics. Curvature terms such as the holomorphic sectional curvature, holomorphic bisectional curvature, and the Chern Ricci curvatures are all introduced. Many examples of Hermitian manifolds with important curvature properties - constant and negative holomorphic sectional curvature, Ricci-flat - are also provided.

Chapter 2 is a survey of the recent developments in the Schwarz lemma. The first section begins with a discussion of the complex Laplacian and the maximum principle. After establishing a general Bochner formula on smooth sections of Hermitian vector bundles proven in [Bro22c], we restrict our attention to holomorphic sections. This yields the Kobayashi–Wu Bochner formula, proven in [KW70]. We then consider holomorphic maps using this formula. Section 2 focuses on the Chern–Lu Schwarz lemma. We discuss developments regarding them, including the extensions by [Roy80], [YZ19], and [BS23]. Several important results are then demonstrated using this formula. This includes the work of [BP23]. The final section discusses the Aubin–Yau Schwarz lemma and recent advances made by [Bro22b] regarding the Schwarz bisectional curvature.

1.1. **Complex Manifolds.** We begin with a reminder of complex differentiability in several complex variables, following [Sha92], [Isa17], [Bro22c]. Throughout, the term *domain* will refer to an open, connected subset of $\mathbb{C}^n$, and *functions* will be understood as scalar-valued maps.

Definition 1.1.1. Let $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}$ be a complex-valued function on a domain Ω. We say that f is $\mathbb{K}$-*differentiable* (either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) at $z \in \Omega$ if

$$f(z + \varepsilon) = f(z) + df(\varepsilon) + O(\varepsilon),$$

for some $\mathbb{K}$-linear function df, and $O(\varepsilon)/|\varepsilon| \to 0$ as $\varepsilon \to 0$. We call the function df the *differential* of f at z.

Example 1.1.2. Consider the function $f(z) = |z|^2$. Let $\varepsilon \in \mathbb{C} \setminus \{0\}$. Then, $f(z+\varepsilon) = |z|^2 + z\overline{\varepsilon} + \overline{z}\varepsilon + |\varepsilon|^2$. We observe that $O(\varepsilon) = |\varepsilon|^2$, and hence $O(\varepsilon)/|\varepsilon| \to 0$ as $\varepsilon \to 0$. Further, we see that $df(\varepsilon)$ is $\mathbb{R}$-linear. However, $df(\varepsilon)$ is not $\mathbb{C}$-linear for all $z \in \mathbb{C}$. One may observe this by comparing $df(\sqrt{-1}\varepsilon)$ and $\sqrt{-1}df(\varepsilon)$. The only value of z that satisfies $df(\sqrt{-1}\varepsilon) = \sqrt{-1}df(\varepsilon)$ is $z = 0$. Hence, f is only $\mathbb{C}$-differentiable at 0.

Recall that for a given basis $\{e_1, \dots e_n\}$ of a vector space V, we may construct the dual basis $\{\varepsilon^1, \dots \varepsilon^n\}$ where $\varepsilon^i(e_j) = \delta^i_j$, with $\delta^i_j = 1$ if and only if $i = j$. Let $\{dx^1, \dots, dx^n, dy^1, \dots, dy^n\}$ denote the dual basis for the standard basis vectors of $\mathbb{R}^{2n}$. We introduce the notation

$$dz^\nu := dx^\nu + \sqrt{-1}dy^\nu, \qquad d\overline{z}^\nu := dx^\nu - \sqrt{-1}dy^\nu.$$

We refer to $z^\nu = x^\nu + \sqrt{-1}y^\nu$ and $\overline{z}^\nu = x^\nu - \sqrt{-1}y^\nu$ as *complex coordinates*. If $f : \Omega \to \mathbb{C}$ is $\mathbb{R}$-differentiable, then we can write the differential as

$$df = \sum_{\nu=1}^n \left(\frac{\partial f}{\partial x^\nu} dx^\nu + \frac{\partial f}{\partial y^\nu} dy^\nu \right).$$

In complex coordinates, df may be written as

$$df = \sum_{\nu=1}^n \left(\frac{\partial f}{\partial z^\nu} dz^\nu + \frac{\partial f}{\partial \overline{z}^\nu} d\overline{z}^\nu \right),$$

where

$$\frac{\partial}{\partial z^\nu} := \frac{1}{2}\left(\frac{\partial}{\partial x^\nu} - \sqrt{-1}\frac{\partial}{\partial y^\nu} \right), \qquad \frac{\partial}{\partial \overline{z}^\nu} = \frac{1}{2}\left(\frac{\partial}{\partial x^\nu} + \sqrt{-1}\frac{\partial}{\partial y^\nu} \right).$$

We write

$$\partial f := \sum_{\nu=1}^n \frac{\partial f}{\partial z^\nu} dz^\nu, \qquad \overline{\partial} f := \sum_{\nu=1}^n \frac{\partial f}{\partial \overline{z}^\nu} d\overline{z}^\nu.$$

where ∂ (resp. $\overline{\partial}$) is referred to as the *holomorphic* (resp. *anti-holomorphic*) *Dolbeault operator*. Observe that since $df(\sqrt{-1}w) = \sqrt{-1}\partial f(w) - \sqrt{-1}\overline{\partial} f(w)$ and $\sqrt{-1}df(w) = \sqrt{-1}\partial f(w) + \sqrt{-1}\overline{\partial} f(w)$, $\mathbb{C}$-linearity implies that $\overline{\partial} f = 0$. These observations are summarised in the following well-known result:

Theorem 1.1.3 (Cauchy–Riemann Equations)**.** Let $f : \Omega \to \mathbb{C}$ be a function which is $\mathbb{R}$-differentiable at a point $z \in \Omega$, where Ω is a domain. Then, f is $\mathbb{C}$-differentiable at z if and only if the *Cauchy–Riemann* equations are satisfied, i.e.,

$$\overline{\partial} f = 0.$$

Remark 1.1.4. In real coordinates $(x^1, \dots x^n, y^1, \dots, y^n)$, The Cauchy–Riemann equations become a system of partial differential equations given by

$$\frac{\partial u}{\partial x^\nu} = \frac{\partial v}{\partial y^\nu}, \qquad \frac{\partial u}{\partial y^\nu} = -\frac{\partial v}{\partial x^\nu}$$

where $u := \mathrm{Re}(f)$, $v := \mathrm{Im}(f)$. We note that for $n > 1$, this system is over determined.

Definition 1.1.5. A function $f : \Omega \to \mathbb{C}$ is said to be *holomorphic at a point* $z \in \Omega$ if it is $\mathbb{C}$-differentiable in some neighbourhood of this point.

A function is referred to as *holomorphic on a domain* Ω if it is holomorphic at every point in Ω. If it is holomorphic on all of $\mathbb{C}^n$, then it is called *entire*. As a cautionary remark, complex differentiability at a point does not imply holomorphy, as shown with Example 1.1.2. The definition of holomorphy is readily extendable to maps of the form $f : \mathbb{C}^n \to \mathbb{C}^m$.

Definition 1.1.6. A map $f : \Omega_1 \to \Omega_2$ where $f = (f^1, \dots, f^m), \Omega_1 \subseteq \mathbb{C}^n, \Omega_2 \subseteq \mathbb{C}^m$ is referred to as *holomorphic at* $p \in \Omega_1$ if each f^i is holomorphic at p. We say f is *holomorphic on* Ω_1, denoted $f \in \mathcal{O}(\Omega_1)$, if each f^i is holomorphic for each point in Ω_1.

The notion of holomorphy allows us to consider an equivalence of complex structures.

Definition 1.1.7. Let $\Omega, \widetilde{\Omega}$ be domains in $\mathbb{C}^n$. A *biholomorphism* is a map $f : \Omega \to \widetilde{\Omega}$ which is holomorphic and invertible. An *automorphism* is a biholomorphism from a domain to itself.

We consider the space of *square integrable holomorphic functions* $L^2_{\mathcal{O}}(\Omega) := L^2(\Omega) \cap \mathcal{O}(\Omega)$. This space inherits the L^2 norm in a very natural manner:

$$\|f\|^2_{L^2_{\mathcal{O}}(\Omega)} := \|f\|^2_{L^2(\Omega)} = \int_\Omega |f(z)|^2 \, d\mu$$

where $|\cdot|$ denotes the modulus of f. It is an inner product space, with the inner product defined by

$$\langle f, g\rangle_{L^2_{\mathcal{O}}(\Omega)} := \int_\Omega f(z)\overline{g(z)} d\mu.$$

The estimate $\sup_{z\in K} |f(z)| \leq C(K)\|f\|_{L^2(\Omega)}$, where K is a compact subset of Ω, implies that $L^2_{\mathcal{O}}(\Omega)$ is a Hilbert space. This estimate also implies that the evaluation functional, defined by

$$\mathrm{ev}_z(f) = f(z)$$

is a continuous linear functional. This allows us to leverage the Riesz representation theorem, in that there exists $\eta_z(\zeta) \in L^2_{\mathcal{O}}(\Omega)$ such that

$$\mathrm{ev}_z(f) = \int_\Omega f(\zeta)\overline{\eta_z(\zeta)} \, d\mu(\zeta).$$

We define $K(z, \overline{\zeta}) := \overline{\eta_z(\zeta)}$ as the *Bergman kernel*. On the unit ball $\mathbb{B}^n$, the Bergman kernel takes the form

$$K(z, \overline{\zeta}) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, \overline{\zeta}\rangle)^{n+1}},$$

where $\langle \cdot, \cdot\rangle$ denotes the standard Hermitian inner product on $\mathbb{C}^n$. The Bergman kernel has found a number of uses throughout complex analysis. Most notably, the work of Fefferman [Fef74] utilised the Bergman kernel to analyse biholomorphisms between certain domains. For our purposes, the Bergman kernel furnishes a particularly important Hermitian inner product, called the *Bergman metric*, on a bounded domain. We now explore the relationship between harmonic and holomorphic functions.

Reminder 1.1.8. A $\mathcal{C}^2$-function $f : \Omega \to \mathbb{R}$, where Ω is an open connected subset of $\mathbb{R}^n$, is referred to as *harmonic* if $\Delta f = 0$ i.e., the Laplacian of f vanishes.

In the case of one complex variable, we may consider a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. The Cauchy–Riemann equations then yield,

$$u_{xx} + u_{yy} = (v_y)_x - (v_x)_y = v_{xy} - v_{yx} = 0$$

where the subscripts denote derivatives and $u := \mathrm{Re}(f), v := \mathrm{Im}(f)$. Therefore, the real part of any holomorphic function is a harmonic function. This argument also follows for v, meaning that the real and imaginary parts of a holomorphic function are harmonic. To extend this notion to higher dimensions, we have the following: for a holomorphic function $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}$, we have that $u = \frac{1}{2}(f + \overline{f})$ and $v = \frac{1}{2}(f - \overline{f})$. Since f is holomorphic,

$$\frac{\partial u}{\partial \overline{z}^\nu} = \frac{1}{2}\left(\frac{\partial f}{\partial \overline{z}^\nu} + \frac{\partial \overline{f}}{\partial \overline{z}^\nu}\right) = \frac{1}{2}\overline{\left(\frac{\partial f}{\partial z^\nu}\right)}.$$

It follows that,

$$\frac{\partial^2 u}{\partial z^\mu \partial \overline{z}^\nu} = \frac{1}{2}\frac{\partial}{\partial z^\mu}\overline{\frac{\partial f}{\partial z^\nu}} = 0,$$

This motivates the following definition.

Definition 1.1.9. Let $\Omega \subseteq \mathbb{C}^n$ be a domain. A $\mathcal{C}^2$-function $u : \Omega \to \mathbb{R}$ is called *pluriharmonic* if

$$\partial\overline{\partial} u = 0.$$

Moreover, we call u *plurisubharmonic* (resp. *strictly plurisubharmonic*) at a point $p \in \Omega$ if the complex Hessian

$$\left(\frac{\partial^2 u}{\partial z^i \partial \overline{z}^j}\right)$$

is positive semi-definite (resp. positive definite) at p. We say u is plurisubharmonic (resp. strictly plurisubharmonic) on Ω if it is plurisubharmonic (resp. strictly plurisubharmonic) for all $p \in \Omega$.

Example 1.1.10. As demonstrated in the previous discussion, the real and imaginary parts of a holomorphic function - in one or several complex variables - are pluriharmonic. By converting into real coordinates, we may also observe that harmonic functions are pluriharmonic. If $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}$ is holomorphic, then $\log|f(z)|$ is plurisubharmonic. This is particularly useful for considering the *Bergman function*, defined as $K(z, \overline{z})$. For cases where $K(z, \overline{z}) > 0$, $\log K(z, \overline{z})$ is strictly plurisubharmonic. This occurs when the associated domain Ω to the Bergman kernel is bounded.

Definition 1.1.11. Let $p \in \mathbb{C}^n$ and $r \in (\mathbb{R}_+)^n$, where $\mathbb{R}_+$ denotes the positive real numbers. We denote by

$$\mathbb{D}^n(p, r) := \{z \in \mathbb{C}^n : |z^\nu - p^\nu| < r_\nu\}$$

the *polydisk centered at* p *with polyradius* r. We will write $\partial\mathbb{D}^n(p, r)$ for the boundary of $\mathbb{D}^n(p, r)$ and define the *skeleton* of $\mathbb{D}^n(p, r)$ as

$$\Gamma(p, r) := \partial\mathbb{D}(p^1, r_1) \times \cdots \times \partial\mathbb{D}(p^n, r_n).$$

We denote $\mathcal{C}(\overline{\Omega})$ as the space of continuous functions on the closure of Ω.

Reminder 1.1.12. Cauchy integral formula.

Recall that for a holomorphic function in one variable $f \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ where $\mathbb{D} := \mathbb{D}(p, r)$, we may represent f using the *Cauchy integral formula* [Isa17, p. 78]: given $z \in \mathbb{D}$, we have that

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

This result is vital for proving the analyticity of holomorphic functions, and consequently Liouville's theorem (see, e.g., [Isa17, p. 107]). In several complex variables, this is also the case.

Theorem 1.1.13 (Cauchy Multiple Integral Formula, [Sha92, p. 18])**.** Let $\mathbb{D}^n := \mathbb{D}^n(p, r)$ denote the polydisk centered at $p \in \mathbb{C}^n$ with polyradius r. Let $f \in \mathcal{O}(\mathbb{D}^n) \cap \mathcal{C}(\overline{\mathbb{D}}^n)$. For any point $z \in \mathbb{D}^n$, the value of f at z can be represented as a multiple Cauchy integral

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma} \frac{f(\zeta)}{(\zeta^1 - z^1)\dots(\zeta^n - z^n)} d\zeta^1 \dots d\zeta^n,$$

where Γ denotes the skeleton of $\mathbb{D}^n$.

Proof. For $z = (z^1, \dots, z^n)$, set $w = (z^1, \dots, z^{n-1})$ such that $z = (w, z^n)$. Write $\mathbb{D}_w^{n-1}$ for the image of the projection of $\mathbb{D}^n$ onto the first $(n-1)$ coordinate directions. The function $f(z) = f(w, z^n)$ is holomorphic in z^n at every point of the disk $\mathbb{D}_n := \{|z^n - p^n| < r_n\}$ and is continuous on its closure. Hence, the Cauchy integral formula for holomorphic functions of one complex variable yields

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\mathbb{D}_n} \frac{f(w, \zeta^n)}{\zeta^n - z^n}\, d\zeta^n.$$

If $w' := (z^1, \dots, z^{n-2})$ such that $z = (w', z^{n-1}, z^n)$, the function $f(w, z^n) = f(w', z^{n-1}, z^n)$ is holomorphic on $\mathbb{D}_{n-1} := \{|z^{n-1} - p^{n-1}| < r_{n-1}\}$ and continuous on its closure. Hence, we may write

$$f(w, \zeta^n) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\mathbb{D}_{n-1}} \frac{f(w', \zeta^{n-1}, \zeta^n)}{\zeta^{n-1} - z^{n-1}}\, d\zeta^{n-1}.$$

Inserting this into the previous expression, we see that

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^2} \int_{\partial D_n} \frac{1}{\zeta^n - z^n} \left(\int_{\partial D_{n-1}} \frac{f(w', \zeta^{n-1}, \zeta^n)}{\zeta^{n-1} - z^{n-1}}\, d\zeta^{n-1} \right) d\zeta^n.$$

Since f is (jointly) continuous on $\overline{\mathbb{D}^n}$, the Fubini-Tonelli theorem implies that

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^2} \int_{\partial\mathbb{D}_n \times \partial\mathbb{D}_{n-1}} \frac{f(w', \zeta^{n-1}, \zeta^n)}{(\zeta^{n-1} - z^{n-1})(\zeta^n - z^n)}\, d\zeta^{n-1}\, d\zeta^n.$$

Iterating this argument completes the proof. $\square$

The integral representation of a holomorphic function leads to a number of remarkable results.

Theorem 1.1.14 ([Sha92, p. 19])**.** Let $\mathbb{D}^n := \mathbb{D}^n(p, r)$. We may write $f \in \mathcal{O}(\mathbb{D}^n)$ as a multiple power series

$$f(z) = \sum_{|k|=0}^{\infty} c_k (z-p)^k,$$

where

$$c_k = \frac{1}{(2\pi\sqrt{-1})^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - p)^{k+1}} \, d\zeta$$

Proof. It suffices to assume that $f \in \mathcal{O}(\mathbb{D}^n) \cap \mathcal{C}(\overline{\mathbb{D}^n})$. Indeed, for any $z \in \mathbb{D}^n$, we may choose a smaller polydisk $\mathbb{D}^n(p, r - \varepsilon)$ which is compactly contained in $\mathbb{D}^n$ and work on this smaller polydisk. Now write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - p} \frac{1}{\left(1 - \frac{z^1 - p^1}{\zeta^1 - p^1}\right) \cdots \left(1 - \frac{z^n - p^n}{\zeta^n - p^n}\right)} = \frac{1}{\zeta - p} \sum_{k=0}^{\infty} \left(\frac{z-p}{\zeta - p}\right)^k = \sum_{k=0}^{\infty} \frac{(z-p)^k}{(\zeta - p)^{k+1}}.$$

For any $z \in \mathbb{D}^n$, this series converges absolutely and uniformly in ζ on Γ. Multiplying by the continuous (and hence, bounded) function $f(\zeta)/(2\pi\sqrt{-1})^n$ on Γ and integrating term by term over Γ, this completes the proof. $\square$

Theorem 1.1.15 (Fundamental Theorem of Hartogs, [Sha92, p. 28])**.** Let f denote a $\mathbb{C}$-valued function from a domain $\Omega \subseteq \mathbb{C}^n$. If f is holomorphic with respect to each variable, then it is holomorphic as a function of n variables.

Remark 1.1.16. The theorem evidently fails for functions $f : \mathbb{R}^2 \to \mathbb{R}$ as evidenced by the standard example

$$\begin{cases} f(x,y) = \frac{xy}{x^2+y^2}, \\ f(0,0) = 0. \end{cases}$$

Although this function is analytic in each variable, it is not even continuous at the origin.

For a holomorphic function $f \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ which is bounded by $M > 0$, we obtain *Cauchy inequalities* in several complex variables:

$$|c_k| \leq \frac{M}{r^k},$$

where $r^k = r_1^k \dots r_n^{k_n}$. Further observe that if f is entire, then by the Cauchy inequalities where $r_1 = \cdots = r_n = r$, we have $|c_k| \leq \frac{M}{r^{|k|}}$. If $|k| = k_1 + \cdots + k_n > 0$, then the limit $r \to \infty$ implies that $c_k = 0$ i.e., $f(z) = c_0$. This yields a fundamental result in complex analysis.

Theorem 1.1.17 (Liouville's Theorem)**.** A bounded and entire function is constant.

We now consider spaces more general than domains in $\mathbb{C}^n$. Throughout, we assume that X is a connected, Hausdorff, paracompact[2] topological space.

[2] To remind ourselves of paracompactness, let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be an open cover of X. The cover $\mathcal{U}$ is said to be *locally finite* if each point in X admits an open neighbourhood that intersects finitely many elements of $\mathcal{U}$. A *refinement* of $\mathcal{U}$ is an open cover $\mathcal{V}$ such that for all $V_\alpha \in \mathcal{V}$, there exists $U_\alpha \in \mathcal{U}$ such that $V_\alpha \subseteq U_\alpha$. Such a topological space is referred to as *paracompact* if every open cover of X admits a locally finite refinement.

Definition 1.1.18. Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be an open covering of X indexed by the set A. Further, assume that each U_α is homeomorphic to the unit ball $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$. The pairs $(U_\alpha, \varphi_\alpha)$, where $\varphi_\alpha : U_\alpha \to \mathbb{B}^n$ is a homeomorphism, is referred to as a *chart*, and the set of charts $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is called an *atlas* of the covering $\mathcal{U}$.

The existence of an atlas allows us to furnish a set of coordinates on our topological space. Indeed, given a chart $(U_\alpha, \varphi_\alpha)$ and a point $p \in U$, we may identify this point with its image under φ_α. Hence, $(U_\alpha, \varphi_\alpha)$ is usually referred to as a *coordinate chart*, where U_α is a *coordinate domain* and φ_α is a *coordinate map*. We say that φ_α is *centered* at $p \in M$ if $\varphi_\alpha(p) = 0$. For $p \in U_\alpha$ where $(U_\alpha, \varphi_\alpha)$ is a coordinate chart, we say that $\varphi_\alpha(p) \in \mathbb{C}^n$ are the *local coordinates* of p.

Consider coordinate charts in the atlas $\mathcal{A}$ on X, denoted $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), such that $U_\alpha \cap U_\beta \neq \varnothing$. On their overlap, we may consider a new map defined by

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).$$

Such a map $\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}$ is referred to as a *transition map*. These allow us to make sense of regularity on X.

Definition 1.1.19. We say that an atlas $\mathcal{A}$ is *holomorphic* if for any two charts on overlapping coordinate domains, their transition map is holomorphic.

To eliminate dependence on any particular holomorphic atlas, we consider two holomorphic atlases $\mathcal{A}$ and $\mathcal{B}$ on X to be *equivalent* if $\mathcal{A} \cup \mathcal{B}$ also forms a holomorphic atlas. It is straightforward to check that this defines an equivalence relation, meaning that we can consider the equivalence class of a holomorphic atlas. Endowing a topological space with such an equivalence class ensures that the transition maps between $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$ and $(V_\beta, \psi_\beta) \in \mathcal{B}$ are holomorphic as well. This approach defines a notion of holomorphy that is intrinsic to X and independent of any specific choice of coordinates. An equivalence class of a holomorphic atlas is referred to as a *complex structure*.

Definition 1.1.20. A *complex manifold* X is a Hausdorff, paracompact topological space equipped with a complex structure. We define the *complex dimension* of X, denoted $\dim_{\mathbb{C}}(X)$, to be the (complex) dimension of the balls which each coordinate domain U_α is homeomorphic to.

Remark 1.1.21. The naming convention for local coordinates changes slightly if X is equipped with a complex structure $\mathcal{A}$. Let (U, φ) be a chart on X and $(z^1, \dots, z^n)$ denote the usual coordinates on $\mathbb{C}^n$. If $p \in X$ is a point, then the entries z^i in $\varphi(p) = (z^1, \dots, z^n)$ are called the *holomorphic coordinates* of p with respect to (U, φ).

Consider an atlas of a topological space M for which each element of the open cover is homeomorphic to $\mathbb{B}^n \subseteq \mathbb{R}^n$, not $\mathbb{C}^n$. Repeating the above construction, but instead imposing that the transition maps be infinitely differentiable, we obtain a *smooth atlas* on M. This yields the following construction.

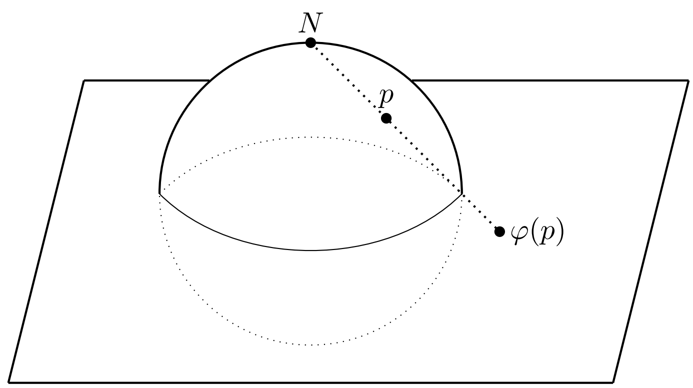
Definition 1.1.22. A *smooth manifold* M is a Hausdorff, paracompact topological space equipped with a *smooth structure* i.e., an equivalence class of smooth atlases. We define the *real dimension* of M, denoted $\dim_{\mathbb{R}}(M)$, to be the (real) dimension of the balls which each coordinate domain U_α is homeomorphic to.

We write M to denote a smooth manifold while X denotes a complex manifold. Given a coordinate chart $(U_\alpha, \varphi_\alpha)$ on M and $p \in M$, we refer to $\varphi_\alpha(p) = (x^1, \dots, x^n)$ as the *local coordinates* of p. Throughout this project, the notation of (x^i) for local coordinates of a smooth chart is used.

All holomorphic maps are analytic and, in particular, smooth. This means that a holomorphic atlas is also a smooth atlas and therefore defines a smooth structure on X. Therefore, every complex manifold is also a smooth manifold in a canonical way. This identification also admits the identity $\dim_{\mathbb{C}}(X) = 2\dim_{\mathbb{R}}(X)$.

Example 1.1.23. A trivial example of a complex manifold is $\mathbb{C}^n$, where it may be covered by a single chart: $(\mathbb{C}^n, \mathrm{Id})$. The identity is holomorphic, meaning that $\mathbb{C}^n$ is a complex manifold.

Example 1.1.24. A complex manifold of (complex) dimension 1 is referred to as a *Riemann surface*. An example of such a manifold is the Riemann sphere $\mathbb{S}^2$. The atlas we consider is furnished by the *stereographic projection*: consider $\mathbb{S}^2$ as a subset of $\mathbb{R}^3$. Our coordinate chart φ then projects from the *north pole* $N = (0, 0, 1)$ onto equatorial plane, which we identify with $\mathbb{C}$.



A similar construction holds for a projection from the *south pole*, which we denote $S := (0, 0, -1)$. This gives us the following maps.

$$\varphi : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}, \qquad \varphi(x, y, z) = \frac{x + \sqrt{-1}y}{1 + z}$$

$$\psi : \mathbb{S}^2 \setminus \{S\} \to \mathbb{C}, \qquad \varphi(x, y, z) = \frac{x - \sqrt{-1}y}{1 + z}$$

The maps φ and ψ are then coordinate charts which cover $\mathbb{S}^2$. The composition $\varphi \circ \psi^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ evaluates to $\varphi \circ \psi^{-1}(z) = \frac{1}{z}$, which is holomorphic on $\mathbb{C} \setminus \{0\}$. Hence, $\mathbb{S}^2$ is a complex manifold.

Example 1.1.25. Consider some n-dimensional vector space V over field $\mathbb{K}$. The *projectivisation of* V, denoted by $\mathbb{P}(V)$, is the set of all equivalence classes under the relation $\sim$ defined by $v_1 \sim v_2$ if and only if there exists $\lambda \in \mathbb{K} \setminus \{0\}$ such that $v_1 = \lambda v_2$. In other words, $\mathbb{P}(V)$ is the set of all 1-dimensional subspaces of V. The most notable examples of these spaces are $\mathbb{P}(\mathbb{R}^n)$ and $\mathbb{P}(\mathbb{C}^n)$, known respectively as the *real projective space* and *complex projective space*. The latter holds particular significance in this project, such that we denote it by $\mathbb{P}^n := \mathbb{P}(\mathbb{C}^{n+1})$.

We now show that $\mathbb{P}^n$ is a complex manifold. To endow $\mathbb{P}^n$ with a complex structure, let $\widetilde{U}_i := \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z^i \neq 0\}$, and let $U_i := \pi(\widetilde{U}_i)$. Define $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i([z^1, \ldots, z^{n+1}]) = \left(\frac{z^1}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots, \frac{z^{n+1}}{z^i}\right).$$

Considering the transition map $\varphi_j \circ \varphi_i^{-1}$ (where we assume $i > j$ WLOG), we obtain

$$\varphi_j \circ \varphi_i^{-1}(z^1, \ldots, z^n) = \left(\frac{z^1}{z^j}, \cdots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^{i-1}}{z^j}, \frac{1}{z^j}, \frac{z^{i+1}}{z^j}, \ldots, \frac{z^n}{z^j}\right).$$

This composition of charts is holomorphic. Hence, it determines a complex structure on $\mathbb{P}^n$. The coordinates $[z^0 : \cdots : z^n]$ are referred to as *homogeneous coordinates*.

More examples of complex manifolds may be proliferated out of previous complex manifolds.

Example 1.1.26. Given complex manifolds $M_1, \ldots, M_k$ of dimensions $n_1, \ldots, n_k$ respectively, we may consider the *product manifold*, given by $M := M_1 \times \cdots \times M_k$. The dimension of M is given by the sum of each $\dim_{\mathbb{C}}(M_i)$, and its complex structure is provided by products of holomorphic coordinate charts.

Example 1.1.27. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{C}^n$. The complex ball $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$ admits a complex structure given by the chart $(\mathbb{B}^n, \mathrm{Id}_{\mathbb{B}^n})$, where $\mathrm{Id}_{\mathbb{B}^n}$ denotes the identity map on $\mathbb{B}^n$. More generally, any open subset of $\mathbb{C}^n$ is a complex manifold.

Example 1.1.28. Let X denote a complex manifold and $U \subseteq X$ be an open subset. An obvious complex structure on U is then given by an atlas consisting of all holomorphic charts on X whose domains are contained in U. This atlas is non-empty, as all points on U are contained in some coordinate chart (W, φ). If we define $V := U \cap W$, then $(V, \varphi|_V)$ is a holomorphic chart on U. Further, all transition maps between coordinate domains are holomorphic by construction. Manifolds of this type are referred to as *open complex submanifolds*.

We provide examples of smooth manifolds that admit no complex structures.

Non-Example 1.1.29. Not all smooth manifolds admit complex structures. In odd dimensions, this is impossible, and even in even dimensions there are smooth manifolds that admit no complex structures - for instance, $\mathbb{S}^4$ cannot be given a complex structure. In fact, a theorem of Kirchoff [Kir47] asserts that this extends to all spheres of dimension 8 or higher.

Remark 1.1.30. There is a unique complex structure on $\mathbb{P}^1$ and $\mathbb{P}^2$, meaning any complex manifold diffeomorphic to $\mathbb{P}^n$ is biholomorphic to $\mathbb{P}^n$ for $n = 1$ or $n = 2$ [Yau77]. For $n = 3$, the uniqueness of the complex structure on $\mathbb{P}^3$ is tied to whether $\mathbb{S}^6$ can be endowed with a complex structure. The only spheres that possibly admit complex structures are $\mathbb{S}^2$ (which corresponds to $\mathbb{P}^1$) and $\mathbb{S}^6$. Although $\mathbb{S}^2$ is known to be a complex manifold, it remains an open problem whether $\mathbb{S}^6$ admits a holomorphic atlas.

Remark 1.1.31. By a theorem of Whitney, any smooth even (real) dimensional manifold admits an analytic structure i.e., the transition maps are analytic [Whi36]. This extension however does not hold for holomorphic atlases, $\mathbb{S}^4$ being a prominent example as described above. Hence, smooth and even dimensional does not necessarily imply complex.

The regularity of an atlas - either smooth or holomorphic - allows us to consider the regularity of maps between manifolds.

Definition 1.1.32. Let X and Y be a complex manifold with charts $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) respectively. A map $f : X \to Y$ is *holomorphic* if the composite map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U_\alpha) \subset \mathbb{C}^n$.

One may also consider the above in the smooth case, where we refer to a map $f : X \to Y$ as *smooth* if the coordinate representation of f is also smooth. The space of all holomorphic functions on X is denoted $\mathcal{O}(X)$. If a map $f : X \to Y$ is holomorphic, then we write $f \in \mathcal{O}(X, Y)$. In the smooth case, smooth $\mathbb{K}$-valued functions (where $\mathbb{K}$ is a field) are denoted by $\mathcal{C}^\infty(X, \mathbb{K})$ and smooth maps are denoted $\mathcal{C}^\infty(X, Y)$. Maps between manifolds also allow us to define an equivalence of smooth and holomorphic structures, as we see with the following.

Definition 1.1.33. Let $f : X \to Y$ denote a map between complex manifolds. We say that f is a *biholomorphism* if it is holomorphic, invertible, and has a holomorphic inverse. If there exists a biholomorphism between X and Y, we say that X and Y are *biholomorphic*, denoted by $X \simeq Y$.

On the other hand, smooth structures are preserved via a *diffeomorphism* i.e., a smooth, invertible map with a smooth inverse. A biholomorphism from a complex manifold to itself is referred to as an *automorphism*. It is easy to check that the set of automorphisms on a complex manifold X form a group under composition, which we denote $\mathrm{Aut}(X)$, called the *automorphism group*. A familiar example is provided by $\mathrm{Aut}(\mathbb{D})$, furnished by rotations and Möbius transformations of the disk. That is, maps of the form

$$e^{\sqrt{-1}\theta}\frac{\alpha - z}{1 - \overline{\alpha}z},$$

for $\theta \in \mathbb{R}, \alpha \in \mathbb{D}$. A particularly important subclass of complex manifolds arise when endowed with a group structure.

Definition 1.1.34. A *complex Lie group* is complex manifold endowed with a group structure such that the composition and inversion maps are holomorphic.

Example 1.1.35. Let V denote an n-dimensional vector space over $\mathbb{C}$, considered as an abelian group. A *lattice* Λ is a discrete, additive subgroup $\Lambda \subseteq V$ generated by a set of $2n$ vectors, which are linearly independent over $\mathbb{R}$. Then, V/Λ forms a (complex) n-dimensional complex Lie group, referred to as a *complex torus*. To show it is diffeomorphic to the n-torus, we may consider the isomorphism of vector spaces $A : \mathbb{R}^{2n} \to V$, given by $A(c^i e_i) = c^i v_i$, where e_i denotes the standard basis vectors in $\mathbb{R}^{2n}$. If we consider the projection maps $\pi_{\mathbb{R}^{2n}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and $\pi_V : V \to V/\Lambda$, we see that A is constant on the fibres of $\pi_{\mathbb{R}^{2n}}$. Similarly, A^{-1} is constant on the fibres of π_V. Hence, $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is diffeomorphic to V/Λ (see, e.g., [Lee13, p. 91]). Observing that $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is diffeomorphic to the n-fold product $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, we obtain our result.

Remark 1.1.36. The above demonstrates that all complex tori are diffeomorphic to each other regardless of the lattice. Not all complex tori are biholomorphic however.

Example 1.1.37. Consider the group Γ generated by the map $z \to \frac{1}{2}z$. The manifold H yielded by quotienting $\mathbb{C}^n \setminus \{0\}$ by Γ is then referred to as a *primary Hopf manifold*. It may also be shown that $H = \mathbb{C}^n \setminus \{0\}/\Gamma$ is diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ (see e.g., [Lee24, p. 11]).

Example 1.1.38. A large number of complex Lie groups are furnished by the well-known theorem of Bochner–Montgomery [BM47], which states that the automorphism group of a compact complex manifold is a complex Lie group.

In complex geometry, it is a worthy goal to understand all compact complex manifolds up to biholomorphism. This story is complete in complex dimension 1 with the *Riemann–Koebe uniformisation theorem*; given a compact Riemann surface Σ with genus g, we have that

(i) $g = 0 \iff \Sigma \simeq \mathbb{P}^1$
(ii) $g = 1 \iff \Sigma \simeq \mathbb{C}/\Lambda$
(iii) $g \geq 2 \iff \Sigma \simeq \mathbb{D}/\Gamma$, where Γ is a discrete subgroup of $\mathrm{Aut}(\mathbb{D})$ acting freely.

The classification of all compact complex surfaces ($\dim_{\mathbb{C}} = 2$) was achieved by Enriques–Kodaira [Enr49], [Kod63], [Kod68]. While classical proofs of the uniformisation theorem largely hinge on harmonic analysis, the work of Enriques–Kodaira is build on the study of *holomorphic curves* i.e., holomorphic maps from Riemann surfaces to complex manifolds. We now discuss several important examples of holomorphic curves.

Example 1.1.39. Let X denote a complex manifold. A curve $f : \mathbb{C} \to X$ is referred to as *entire* if it is non-constant and holomorphic on all of $\mathbb{C}$. Further, X is said to be *Brody hyperbolic* if it admits no entire curves [Bro78]. A classical example of a Brody hyperbolic manifold is the unit disk $\mathbb{D} \subset \mathbb{C}$; indeed, $\mathbb{D}$ admits no entire curves thanks to Liouville's theorem.

Example 1.1.40. A *rational curve* is a non-constant holomorphic map from $\mathbb{P}^1$ to a complex manifold X. The *Fermat hypersurfaces* of degree $d \in \mathbb{N}$, given by

$$\mathrm{Fer}_d := \{[z_0 : \cdots : z_n] \in \mathbb{P}^n : z_0^d + \cdots + z_n^d = 0\}$$

support rational curves of the form $f_\zeta : \mathbb{P}^1 \to \mathrm{Fer}_d$, where $f_\zeta(u, v) := (u, v, \zeta u, \zeta v, 0 \ldots, 0)$ and $\zeta^d = -1$. A complex manifold X is *rationally connected* if any two points on X lie in the image of some rational curve. As $\mathbb{P}^1$ is the one-point compactification of $\mathbb{C}$, if X admits a rational curve, then it admits an entire curve. Consequently, rationally connected manifolds are not Brody hyperbolic. Rational curves play a central role in many areas of complex geometry, making them fundamentally important (see e.g., [BHPVdV04], [Rei87], [Voi03]).

Example 1.1.41. A non-constant holomorphic curve from the torus (complex dimension 1) to a complex manifold is referred to as an *elliptic curve*.

It is helpful to examine constructions on complex manifolds that involve only the smooth structure, as they may be naturally extended to the complex setting through complexification. As with regular surfaces, it is natural to consider vectors that are tangent to a manifold at a point. A complex manifold however does not necessarily lie in some ambient space, meaning we must define tangent vectors intrinsically.

Definition 1.1.42. Let X be a complex manifold, and $p \in X$ be a point. A linear map $V : \mathcal{C}^\infty(X, \mathbb{R}) \to \mathcal{C}^\infty(X, \mathbb{R})$ is called a *derivation at p* if it satisfies the Leibniz rule:

$$V(fg) = f(p)V(g) + g(p)V(f).$$

The space of all derivations at p, denoted T_pX is referred to as the *tangent space* of X at p.

A basis at $T_a\mathbb{R}^{2n}$ is given by the $2n$ derivations $\frac{\partial}{\partial x^1}|_a, \ldots, \frac{\partial}{\partial x^{2n}}|_a$ where

$$\left.\frac{\partial}{\partial x^i}\right|_a f = \frac{\partial f}{\partial x^i}(a)$$

for $1 \leq i \leq 2n$.

The above definition does not rely on X living in some ambient space. Further, we observe that T_pX forms a vector space since derivations are linear. Given a smooth map $f : X \to Y$, a linear approximation of f is induced between tangent spaces. This map is called the differential:

Definition 1.1.43. Let $f : X \to Y$ be a smooth map between complex manifolds. At a point $p \in X$, the induced map between the tangent spaces $df_p : T_pX \to T_{f(p)}Y$ is referred to as the *differential* of f at p, and is given by

$$df_p(V)(g) = V(g \circ f)$$

where $V \in T_pX, g \in \mathcal{C}^\infty(X, \mathbb{R})$, and $g \in \mathcal{C}^\infty(Y, \mathbb{R})$.

Given a diffeomorphism $f : X \to Y$, its differential df_p at $p \in X$ is a vector space isomorphism. Suppose that $\dim_{\mathbb{R}}(X) = 2n$ and admits local coordinates $(x^1, \ldots x^{2n})$ under the chart (U, φ). Since φ may be thought of as a diffeomorphism from U to $\varphi(U)$, we obtain that $T_pU \simeq T_{\varphi(p)}\varphi(U)$

as vector spaces. Further, as the differential of the inclusion map $U \hookrightarrow X$ is also a vector space isomorphism (see, e.g., [Lee13]), we have $T_pX \simeq T_{\varphi(p)}\mathbb{R}^{2n}$. We then see that T_pX is $2n$-dimensional, and admits the basis $\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^{2n}}|_p$.

Definition 1.1.44. Let $f : X \to Y$ be a smooth map between complex manifolds. We say that f is

(i) an *immersion* if df_p is injective for all $p \in X$.
(ii) a *submersion* if df_p is surjective for all $p \in X$.
(iii) an *embedding* if f is an immersion and homeomorphic onto its image.

Definition 1.1.45. Let X and Y be complex manifolds. We say that X is a *complex submanifold* of Y if there exists a holomorphic embedding $X \hookrightarrow Y$.

Remark 1.1.46. All complex manifolds admit a smooth embedding into $\mathbb{R}^n$ for some $n \in \mathbb{N}$ by the Whitney embedding theorem [Whi44]. If we instead require the embedding to be holomorphic into $\mathbb{C}^n$, the theorem fails in a spectacular fashion: if X is a compact, complex manifold, then there is no holomorphic embedding into $\mathbb{C}^n$ for any $n \in \mathbb{N}$, unless X is a point. Indeed, if such an embedding exists, the coordinate functions on $\mathbb{C}^n$ would restrict to holomorphic functions on a compact set. Hence, they are all constant by the maximum modulus principle. A holomorphic analogue for the Whitney embedding theorem is the Kodaira embedding theorem [Kod54], which allows complex manifolds that are compact (along with some other topological conditions) to be embedded into $\mathbb{P}^n$ for some $n \in \mathbb{N}$.

The complex manifolds for which a holomorphic embedding into $\mathbb{C}^n$ or $\mathbb{P}^n$ exists form important subclasses.

Definition 1.1.47. A complex manifold S is said to be *Stein* if there is a holomorphic embedding $S \hookrightarrow \mathbb{C}^n$ for some $n \in \mathbb{N}$.

Example 1.1.48. An open subset of $\mathbb{C}^n$ which is non-compact is a Stein manifold; the unit ball $\mathbb{B}^n$, $\mathbb{C}^n \setminus \{0\}$ and the *upper half-plane* $\mathbb{H} := \{x + \sqrt{-1}y : y > 0\}$ are all familiar examples. Further, a theorem of Behnke–Stein [BS47] asserts that all non-compact Riemann surfaces are Stein manifolds. In particular, $\mathbb{C}/\Lambda$ with a point removed is a Stein manifold.

Definition 1.1.49. A complex manifold X is said to be *projective* if there is a holomorphic embedding $X \hookrightarrow \mathbb{P}^n$ for some $n \in \mathbb{N}$.

Example 1.1.50. We have already encountered an example of a projective manifold with the Fermat hypersurfaces. These manifolds fall into a larger class of projective manifolds which arise zero sets of homogeneous polynomials in projective space without singularities, known as *smooth projective varieties.* In fact, a well-known theorem of Chow [Cho49] asserts that projective manifolds are equivalent to smooth projective varieties. As they are outside the scope of this thesis, we invite the reader to consult [GH94] for a thorough treatment of the topic.

1.2. **Holomorphic Vector Bundles.** The following class of submersions will prove vital throughout the project.

Definition 1.2.1. Let X be a complex manifold, and E be a smooth manifold. We refer to $\pi : E \to X$ as a *smooth vector bundle* of rank k if each fibre $E_p := \pi^{-1}(p) \simeq \mathbb{R}^k$, and for all $p \in X$, and in every neighbourhood of p, there exists some open neighbourhood U and a map $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that φ is a diffeomorphism.

A rank k vector bundle $\pi : E \to X$ is referred to as *topologically trivial* if E is homeomorphic to $X \times \mathbb{R}^k$. Hence, in the above definition, each φ is referred to as a *trivialising chart*. The map $\pi : E \to X$ is called the *projection*. Namely, it is from the *total space* E to the *base* X.

A vector bundle allows us to associate each point p on the base X with an element in the fibre of p. This association is referred to as a *section*: a continuous map $\sigma : U \subseteq X \to E$ such that $\pi \circ \sigma = \mathrm{Id}$. If σ is a smooth map, then it is called a *smooth section*. The space of all smooth sections of a vector bundle $E \to X$ is denoted $\Gamma(E)$. Given a collection of smooth sections $(\sigma_1, \dots, \sigma_k)$ over $U \subseteq X$, we say that it is a (*local*) *frame* for E over U if $(\sigma_1(p), \dots, \sigma_k(p))$ is a basis for E_p for each $p \in U$.

Example 1.2.2. Let X be a complex manifold with $\dim_{\mathbb{R}}(X) = 2n$. We define the *tangent bundle* TX by

$$TX := \bigsqcup_{p \in M} T_p X.$$

The fibre at p is T_pX, which is isomorphic to $\mathbb{R}^{2n}$ as shown prior. Further, the tangent bundle itself is a $4n$ (real) dimensional smooth manifold, as we may leverage the smooth manifold structure on X: given a coordinate chart (U, φ) on X, with local coordinates (x^i), we define the charts

$$\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{4n}, \qquad \widetilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = (\varphi(p), v^1, \dots, v^{2n})$$

where $p \in U$ and $v^i \in \mathbb{R}^{2n}$. We may define the trivialising charts in a similar fashion, given by $\widetilde{\varphi}(v^i \frac{\partial}{\partial x^i}|_p) = (p, v^1, \dots, v^{2n})$. It is trivial to check that the conditions for TX to be a smooth vector bundle are satisfied. The smooth sections of TX are called *smooth vector fields*.

Example 1.2.3. The *cotangent bundle* T^*X is given by

$$T^*X := \bigsqcup_{p \in X} T_p^* X$$

where T_p^*X denotes the dual vector space of T_pX. Given a chart (U, φ) on X with local coordinates (x^i), T^*X is locally framed by $(dx^1, \dots, dx^{2n})$. Smooth sections of T^*X are referred to as *1-forms*.

Recall that we may proliferate vector spaces using the usual operations of the direct sum, the tensor product, and dualising. These operations are easily extendable to vector bundles.

Definition 1.2.4. Let $E, F \to X$ be smooth vector bundles over a complex manifold X. We define the

(i) *direct sum bundle* $E \oplus F \to X$ as the vector bundle with fibres $E_p \oplus F_p$ for each $p \in X$.
(ii) *tensor bundle* $E \otimes F \to X$ as the vector bundle with fibres $E_p \otimes F_p$ for each $p \in X$.
(iii) *dual bundle* E^* as the vector bundle with fibres $(E_p)^*$ for each $p \in X$.

The following vector bundle will prove important throughout this project.

Definition 1.2.5. Let $f : X \to Y$ be a smooth map between complex manifolds, and $E \to Y$ be a smooth vector bundle. The *pullback bundle* is the vector bundle over X defined as

$$f^*E := \bigsqcup_{p \in X} E_{f(p)} = \{(p, e) \in X \times E : f(p) = \pi(e)\}.$$

Reminder 1.2.6. Tensors

Let V be an n-dimensional vector space over $\mathbb{R}$. A *covariant k-tensor* is an element of the k-fold tensor product $V^* \otimes \cdots \otimes V^*$ i.e., a real-valued, multi-linear function α given by

$$\alpha : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}.$$

Conversely, a *contravariant k-tensor* is an element of the k-fold tensor product $V \otimes \cdots \otimes V$ i.e., a real-valued, multi-linear function ω given by

$$\omega : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \to \mathbb{R}.$$

More generally, given $k, \ell \in \mathbb{Z}_{\geq 0}$, we say that α is of *type* (k, ℓ) if it is an element of

$$\underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell \text{ times}}$$

or put simply, it is a real-valued, multi-linear function with ℓ inputs in V, and k inputs in V^*. The space of all tensors of type (k, ℓ) is denoted $T^{(k,\ell)}(V)$. Given a basis $\{e_i\}_{i=1}^n$ for V with corresponding dual basis $\{\varepsilon^i\}_{i=1}^n$, we may define a basis for $T^{(k,\ell)}(V)$ given by

$$e_{j_1} \otimes \cdots \otimes e_{j_\ell} \otimes \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}.$$

Hence, we may write any tensor α of type (k, ℓ) as

$$\alpha = \alpha^{j_1 \ldots j_\ell}{}_{i_1 \ldots i_k} e_{j_1} \otimes \cdots \otimes e_{j_\ell} \otimes \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$$

where $\alpha^{j_1 \ldots j_\ell}{}_{i_1 \ldots i_k} = \alpha(\varepsilon^{j_1}, \ldots, \varepsilon^{j_\ell}, e_{i_1}, \ldots, e_{i_k})$. The *trace* of α can also be considered, defined as the $(k-1, \ell-1)$ tensor

$$\begin{aligned}\operatorname{tr}_{i_p j_q}(\alpha) = \alpha^{j_1 \ldots j_{q-1} m j_{q+1} \ldots j_\ell}{}_{i_1 \ldots i_{p-1} m i_{p+1} \ldots i_k} e_{j_1} &\otimes \cdots \otimes e_{j_{q-1}} \otimes e_{j_{q+1}} \otimes \cdots \otimes e_{j_\ell} \\ &\otimes \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_{p-1}} \otimes \varepsilon^{i_{p+1}} \otimes \cdots \otimes \varepsilon^{i_k}.\end{aligned}$$

In the above case, we say that α has been *contracted* over the indices i_p, j_q. We say that covariant k-tensor α is *alternating* if interchanging any of the two inputs changes the sign of the tensor. An example of such a tensor is the determinant, where interchanging two columns changes its sign. We denote the space of all covariant alternating k-tensors on V by $\Lambda^k(V^*)$, referred to as the *kth exterior power* of V^*. If permuting any two inputs of a tensor does not change its sign, it is referred to as *symmetric*. An example of such a tensor is an inner product on a vector space V, which is a symmetric covariant 2-tensor.

Extending the notion of an alternating tensor to a manifold, we may define the bundle

$$\Lambda^k(T^*X) := \bigsqcup_{p \in X} \Lambda^k(T_p^*X).$$

Considering the sections of this bundle, we obtain the following.

Definition 1.2.7. Let X be a complex manifold. We refer to smooth sections of $\Lambda^k(T^*X)$ as *k-forms*. The space of all k-forms on X is denoted by $\Omega^k(X)$.

The notion of a k-form may also be extended to consider outputs in the total space of a vector bundle $E \to X$. Hence, we define an *E-valued k-form* as a smooth section of $E \otimes \Lambda^k(T^*X)$. The space of all such forms is denoted $\Omega^k(X, E)$.

Definition 1.2.8. Let X be a complex manifold. The *(k,ℓ) tensor bundle* $T^{(k,\ell)}(X) \to X$ is given by

$$T^{(k,\ell)}(X) := \bigsqcup_{p \in X} T^{(k,\ell)}(T_pX).$$

Sections of $T^{(k,\ell)}(M)$ are referred to as *(k,ℓ)-tensor fields*, or simply *(k,ℓ)-tensors*.

Covariant tensor fields play a crucial role in this project. Let $f : X \to Y$ be a smooth map and α be a covariant k-tensor field on Y. We can construct a corresponding covariant k-tensor field on X by defining the *pullback* of α by f, denoted $f^*\alpha$ and given by

$$(f^*\alpha)_p(v_1, \ldots, v_k) = \alpha(df_p(v_1), \ldots df_p(v_k))$$

for each $p \in X$ and $v_1, \ldots, v_k \in T_pY$.

Reminder 1.2.9. The Exterior Algebra. We say that an algebra A is *graded* if it admits a decomposition into the direct sum $\bigoplus_{k\in\mathbb{Z}}^{\infty} A^k$, where A^k are algebras such that $A^kA^\ell \subseteq A^{k+\ell}$. The *exterior algebra* $\Lambda^\bullet(V^*)$ is defined as the vector space

$$\Lambda^\bullet(V^*) := \bigoplus_{k=0}^{n} \Lambda^k(V^*).$$

When equipped with the wedge product, $\Lambda^\bullet(V^*)$ admits the structure of a graded algebra. Indeed, we are afforded this decomposition by identifying $A^k = \Lambda^k(V^*)$ for $0 \leq k \leq n$, and $A^k = \{0\}$ otherwise.

The grading of the exterior algebra may be extended to the space of forms:

$$\Omega^\bullet(X) = \bigoplus_{k=0}^{n} \Omega^k(X).$$

Definition 1.2.10. Let X be a complex manifold. The *exterior derivative* is defined as the unique $\mathbb{R}$-linear mapping

$$d : \Omega^\bullet(X) \to \Omega^\bullet(X)$$

such that

(i) $d : \Omega^k(X) \to \Omega^{k+1}(X)$
(ii) $d(f) = df$ (the differential of f) for $f \in \mathcal{C}^\infty(X, \mathbb{R})$
(iii) If $\sigma \in \Omega^k(X)$, $\tau \in \Omega^\ell(X)$, then

$$d(\sigma \wedge \tau) = d\sigma \wedge \tau + (-1)^k \sigma \wedge d\tau$$

(iv) $d^2 = 0$

Let (x^i) denote smooth coordinates on a complex manifold X. We may write a k-form ω in these coordinates as $f dx^{i_1} \wedge \ldots dx^{i_k}$, where $f \in \mathcal{C}^\infty(X, \mathbb{R})$ and $1 \leq i_p \leq \dim_{\mathbb{R}}(X)$ for $1 \leq p \leq k$. The definition of the exterior derivative then yields that

$$d\omega = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \ldots dx^{i_k}.$$

The exterior derivative on complex manifolds induces a *cochain complex* on the space of forms i.e., $(\Omega^\bullet(X), d)$ forms the sequence

$$\mathcal{C}^\infty(X, \mathbb{R}) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X) \xrightarrow{d} \ldots$$

such that $d^2 = 0$. Cochain complexes give rise to cohomologies, where in this case we recover the *de Rham cohomology.* The *pth de Rham cohomology group* is

$$H^p_{dR}(X,\mathbb{R}) := \frac{\ker\{d : \Omega^p(X) \to \Omega^{p+1}(X)\}}{\operatorname{im}\{d : \Omega^{p-1}(X) \to \Omega^p(X)\}}.$$

For $k \in \mathbb{N}$, the *kth Betti number* is defined as $b_k(X) := \dim_{\mathbb{R}} H^p_{dR}(X,\mathbb{R})$.

Definition 1.2.11. Let $\alpha \in \Omega^k(X)$, where X denotes a complex manifold. We say α is *closed* if $d\alpha = 0$. If there exists $\beta \in \Omega^{k-1}(X)$ such that $d\beta = \alpha$, then α is referred to as *exact.*

Remark 1.2.12. By the nilpotence property of the exterior derivative ($d^2 = 0$), every exact form is closed. The failure for the converse to hold is measured by the de Rham cohomology group. On the other hand, the Poincaré lemma gives sufficient conditions for a closed differential form to be exact.

We now extend these constructions through complexification. By a theorem of Newlander–Nirenberg [NN57], we may recover the complex structure on a complex manifold X from an endomorphism $J : TX \to TX$ satisyfing $J^2 = -\mathrm{Id}$ and subject to an integrability condition. In particular, the relation $J^2 = -\mathrm{Id}$ yields the eigenvalues of J being $\pm\sqrt{-1}$. This prompts us to *complexify* the tangent bundle:

$$T_{\mathbb{C}}X := \bigsqcup_{p\in X} T_pX \otimes_{\mathbb{R}} \mathbb{C}.$$

That is, $T_{\mathbb{C}}X$ is obtained through complexifying over the fibres. The J map therefore induces a splitting of $T_{\mathbb{C}}X$ into eigenbundles of eigenvalues $\pm\sqrt{-1}$:

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X.$$

We refer to $T^{1,0}X$ (the $+\sqrt{-1}$ eigenbundle) and $T^{0,1}X$ (the $-\sqrt{-1}$ eigenbundle) as the *holomorphic* and *antiholomorphic* tangent bundles respectively. These bundles can be locally framed by coordinate frames $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \bar{z}^i}$ respectively. We may also characterise these bundles by

$$T^{1,0}X = \{v - \sqrt{-1}Jv : v \in TX\}, \qquad T^{0,1}X = \{v + \sqrt{-1}Jv : v \in TX\}.$$

This splitting is also induced on the *complexified cotangent bundle* $T^*_{\mathbb{C}}X$, which decomposes into $\Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X)$, where $\Lambda^{1,0}(X)$ and $\Lambda^{0,1}(X)$ denote the bundles of $(1,0)$-forms and $(0,1)$-forms respectively. A $(1,0)$-form on X is a complex-valued 1-form ξ such that $\xi(v) = 0$ for all $v \in T^{0,1}X$. A $(0,1)$-form η is similarly defined, with $\eta(v) = 0$ for all $v \in T^{1,0}X$. We let $\Lambda^{k,0}(X)$ and $\Lambda^{0,q}(X)$ denote the kth and qth exterior power of $\Lambda^{1,0}(X)$ and $\Lambda^{0,1}(X)$ respectively. We define $\Lambda^{p,q}(X) := \Lambda^{p,0}(X) \otimes \Lambda^{0,q}(X)$. For vector bundles E, F over a manifold of dimension n,

$$\Lambda^k(E \oplus F) = \bigoplus_{i=0}^{n} \Lambda^i(E) \otimes \Lambda^{k-i}(F).$$

The splitting on $T^*_{\mathbb{C}}X$ then reduces the above to

$$\Lambda^k_{\mathbb{C}}(T^*_{\mathbb{C}}X) = \bigoplus_{p+q=k} \Lambda^{p,q}(X)$$

Definition 1.2.13. Let X be a complex manifold. For $0 \le p, q \le n = \dim_{\mathbb{C}} X$, we define *(p,q)-forms* as smooth sections of $\Lambda^{p,q}(X)$. We denote the space of all such forms as $\Omega^{p,q}(X)$.

Definition 1.2.14. Let $\omega \in \Omega^{(p,p)}(X)$ be a real (p,p)-form (i.e., invariant under conjugation). We say ω is *positive* if

$$(-\sqrt{-1})^p \alpha(v_1, \overline{v_1}, \ldots, v_p, \overline{v_p}) > 0$$

for any set of linearly independent (over $\mathbb{C}$) vectors $v_1, \ldots, v_p$.

The space of complex-valued k-forms is afforded a splitting into (p,q)-forms

$$\Omega^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X)$$

where $\Omega^k_{\mathbb{C}}(X) = \Omega^k(X) \otimes_{\mathbb{R}} \mathbb{C}$. Choosing holomorphic coordinates $\{z^i\}$ on X, a (p,q)-form α may be then locally given by

$$\alpha = \sum_{1 \leq i_1 \leq \cdots \leq i_p \leq n} \sum_{1 \leq j_1 \leq \cdots \leq j_q \leq n} f_{i_1 \dots i_p, j_1, \dots, j_q} dz^1 \wedge \dots dz^p \wedge d\overline{z}^1 \cdots \wedge d\overline{z}^q.$$

If the component functions are holomorphic, we refer to α as a *holomorphic (p,q)-form.* The presence of an almost complex structure induces a bigrading on the exterior algebra.

$$\Lambda^\bullet_{\mathbb{C}}(T^*_{\mathbb{C}}X) = \bigoplus_{k=0}^{n} \bigoplus_{p+q=k} \Lambda^{p,q}(X).$$

This bigrading is also afforded to the algebra of complex-valued forms.

$$\Omega^\bullet_{\mathbb{C}}(X) = \bigoplus_{k=0}^{n} \bigoplus_{p+q=k} \Omega^{p,q}(X).$$

We consider the complex-linear extension of the exterior derivative d to $\Omega^\bullet_{\mathbb{C}}(X)$. The complex structure induces a splitting of d into two natural operators.

Definition 1.2.15. Let $d : \Omega^k_{\mathbb{C}}(X) \to \Omega^{k+1}_{\mathbb{C}}(X)$ denote the usual exterior derivative on complex-valued k-forms. We define the *Dolbeault operators* $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ and $\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$ as

$$\partial := \pi^{p+1,q} \circ d, \qquad \overline{\partial} := \pi^{p,q+1} \circ d$$

where $\pi^{p+1,q} : \Omega^{k+1}_{\mathbb{C}}(X) \to \Omega^{p+1,q}(X)$ and $\pi^{p,q+1} : \Omega^{k+1}_{\mathbb{C}}(X) \to \Omega^{p,q+1}(X)$ denote the natural projection maps.

Remark 1.2.16. The Dolbeault operators provide an equivalent definition of holomorphy: a $\mathcal{C}^1$ complex-valued function from a complex manifold $f : X \to \mathbb{C}$ is holomorphic if and only if $\overline{\partial} f$ vanishes identically. This is due to the equation $\overline{\partial} f = 0$ merely being the Cauchy–Riemann equations.

Proposition 1.2.17 ([Lee24, p. 106])**.** The Dolbeault operators support the identity

$$0 = \partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial$$

Proof. First observe that $d = \partial + \overline{\partial}$. Since $d^2 = 0$, we have that $0 = (\partial + \overline{\partial})^2 = \partial^2 + \overline{\partial}^2 + \partial\overline{\partial} + \overline{\partial}\partial$. Since all terms in the sum take values in different vector bundles, they all equate to 0. □

As with the exterior derivative, the antiholomorphic Dolbeault operator also induces a cochain complex on the space of (p,q)-forms, as $\overline{\partial}^2 = 0$ and we have the sequence

$$\Omega^{p,0}(X) \xrightarrow{\overline{\partial}} \Omega^{p,1}(X) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Omega^{p,q}(X) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(X) \xrightarrow{\overline{\partial}} \dots$$

Definition 1.2.18. Let X be a complex manifold and $\overline{\partial}$ denote the Dolbeault operator on $\Omega^{p,q}(X)$. The *(p,q) Dolbeault cohomology group* is given by

$$H^{p,q}_{\overline{\partial}}(X) := \frac{\ker\{\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)\}}{\operatorname{im}\{\overline{\partial} : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X)\}}.$$

We define the *Hodge numbers* as $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X)$

In order to study the geometry of complex manifolds, it becomes apparent we must consider metrics on vector bundles. To this end, we begin with the following.

Definition 1.2.19. Let X be a complex manifold, and E be a smooth manifold. We refer to $\pi : E \to X$ as a *holomorphic vector bundle* of rank k if each fibre $E_p := \pi^{-1}(p)$ is isomorphic to $\mathbb{C}^k$ as vector spaces, and for all $p \in X$, and in every neighbourhood of p, there exists some open neighbourhood U and a map $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that φ is a biholomorphism.

It should be noted that an equivalent definition for a holomorphic vector bundle is that the transition maps for any pair of trivialising charts (on which their domains agree) are holomorphic.

Example 1.2.20. Let X be a complex manifold. The bundle $T^{1,0}X$ is a holomorphic vector bundle. The holomorphic sections of $T^{1,0}X$ are referred to as *holomorphic vector fields*.

Example 1.2.21. Let $\dim_{\mathbb{C}}(X) = n$. A holomorphic vector bundle $\mathcal{L} \to X$ of rank 1 is referred to as a *holomorphic line bundle*. The most important example of a holomorphic line bundle is given to us by the *canonical bundle* $K_X := \Lambda^{n,0}(X)$. The local holomorphic sections of K_X are given by

$$\omega = f(z)dz^1 \wedge \cdots \wedge dz^n$$

where f is a locally defined holomorphic function. In the case of compact Riemann surfaces, the holomorphic sections of K_X is precisely the space of $(1,0)$-forms $\Omega^{(1,0)}(X)$. The canonical bundle is intimately connected with the genus g; using the Riemann–Roch theorem, one may demonstrate that $\dim_{\mathbb{C}} H^0(X, K_X) = g$.

As previously mentioned, considering the holomorphic functions on a complex manifold may be a fruitless endeavour, particularly if the manifold is compact. Hence, it is useful to instead consider sections of K_X. To this end, consider the map $\Phi : X \to \mathbb{P}^n$ for some n, given by

$$x \mapsto [\sigma_0(x) : \cdots : \sigma_n(x)],$$

where $\sigma_0, \ldots, \sigma_n$ are holomorphic sections of K_X. If Φ defines an embedding for some n, we refer to the canonical bundle as *very ample*. Further, we refer to K_X as *ample* if there exists $k \in \mathbb{N}$ such that $\mathcal{L}^{\otimes k}$ is very ample.

Definition 1.2.22. Let $E \to X$ be a complex vector bundle over a complex manifold X. A first-order $\mathbb{C}$-linear differential operator

$$\overline{\partial}^E : \Gamma(E) \to \Omega^{0,1}(X) \otimes \Gamma(E)$$

is said to be a *pseudoholomorphic structure* if it satisfies the following variant of the Leibniz rule:

$$\overline{\partial}^E(f\sigma) = \overline{\partial}f \otimes \sigma + f\overline{\partial}^E(\sigma)$$

where $f \in \mathcal{C}^\infty(X, \mathbb{C})$, $\sigma \in \Gamma(E)$ is a smooth section, and $\overline{\partial} : \mathcal{C}^\infty(X, \mathbb{C}) \to \Omega^{0,1}(X)$ is the Dolbeault operator in Definition 1.2.15. If $(\overline{\partial}^E)^2 = 0$, then $\overline{\partial}^E$ is called a *holomorphic structure* (or a *Dolbeault operator*).

A section σ in a pseudoholomorphic vector bundle $(E, \overline{\partial}^E)$ is called *holomorphic* if $\overline{\partial}^E\sigma = 0$. The space of holomorphic sections of E is denoted by $H^0(E)$. This choice in nomenclature hints at a link between our two notions of "holomorphic" for a vector bundle. Indeed, the following theorem proven by Koszul & Malgrange [KM58] shows that a holomorphic vector bundle and a complex vector bundle with a holomorphic structure actually coincide.

Theorem 1.2.23 (Koszul–Malgrange, [KM58])**.** Let X be a complex manifold. A complex vector bundle $E \to X$ is holomorphic vector bundle if and only if it has a holomorphic structure $\overline{\partial}$.

The proof of Koszul–Malgrange does not contribute further to our discussion of holomorphic vector bundles, therefore we omit it (see [DK90],[BGM71] for details). The Koszul–Malgrange theorem is highly significant for studying holomorphic vector bundles; we now have access to a differential operator that allows us to work independent of the coordinate system in an easy fashion.

1.3. **Hermitian Structures on Holomorphic Vector Bundles.** In the following definition, we let X be a complex manifold and $E \to X$ be a complex vector bundle. We denote $\Omega^1_{\mathbb{C}}(X, E) := \Omega^1(X, E) \otimes_{\mathbb{R}} \mathbb{C}$, where $\Omega^1(X, E)$ is the bundle of E-valued 1-forms. The bundle $\Omega^1_{\mathbb{C}}(X, E)$ is referred to as the *E-valued complex 1-forms.* The complex structure also induces a splitting of this bundle similar to that of (p, q)-forms on the complexified tangent bundle.

Definition 1.3.1. A *connection on* E is a $\mathbb{C}$-linear differential operator $\nabla : \Gamma(E) \to \Omega^1_{\mathbb{C}}(X, E)$ such that for all $f, g \in \mathcal{C}^\infty(X, \mathbb{C})$, $u, v \in \Gamma(T_{\mathbb{C}}X)$ and $\sigma \in \Gamma(E)$

(i) it is $\mathcal{C}^\infty$-linear in the first variable
$$\nabla_{fu+gv}(\sigma) = f\nabla_u\sigma + g\nabla_v\sigma$$
(ii) it satisfies the Leibniz rule:
$$\nabla_u(f\sigma) = u(f)\sigma + f\nabla_u\sigma$$

Given a local coordinate frame $\partial_i := \frac{\partial}{\partial z^i}$, we may consider the *Christoffel symbols* of our connection ∇, defined by the equation $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$. If we use the antiholomorphic coordinate frame $\partial_{\bar{i}} := \frac{\partial}{\partial \bar{z}^i}$, we bar that index i.e., $\nabla_{\partial_{\bar{i}}}\partial_j = \Gamma^k_{\bar{i}j}\partial_k$.

Example 1.3.2. Let ∇ be a connection on $E \to Y$ and $f : X \to Y$ be a smooth map between complex manifolds. The *pullback connection* on f^*E, denoted $f^*\nabla$, is given by
$$f^*\nabla_Z(\omega \circ f) := \nabla_{df(Z)}\omega$$
where $Z \in \Gamma(T_{\mathbb{C}}X)$ and $\omega \in \Gamma(E)$.

Example 1.3.3. Let $E \to X$ be a complex vector bundle with connection ∇. The *dual connection* on E^*, denoted ∇^*, is defined by
$$(\nabla^*_Z\omega)(Y) := X(\omega(Y)) - \omega(\nabla_X Y)$$
where $Z \in \Gamma(T_{\mathbb{C}}X)$, $\omega \in \Gamma(E^*)$ and $Y \in \Gamma(E)$.

Observe that the above example essentially forces the product rule necessary for ∇^* to be a connection.

Example 1.3.4. Given connections ∇^1, ∇^2 on complex vector bundles $E_1, E_2 \to X$ respectively, the *tensor product connection* on $E_1 \otimes E_2$ is given by
$$\nabla_Z(\omega \otimes \eta) := \nabla^1_Z\omega \otimes \eta + \omega \otimes \nabla^2_Z\eta$$
where $\omega \in \Gamma(E_1), \eta \in \Gamma(E_2)$ and $Z \in \Gamma(T_{\mathbb{C}}X)$. Here, we use the identification $\Gamma(E_1 \otimes E_2) = \Gamma(E_1) \otimes \Gamma(E_1)$.

We may split a connection ∇ into its (1,0) and (0,1) parts:
$$\nabla^{1,0} := \pi^{1,0} \circ \nabla, \qquad \nabla^{0,1} := \pi^{0,1} \circ \nabla$$
where $\pi^{1,0} : \Omega^1_{\mathbb{C}}(X, E) \to \Omega^{1,0}(X, E)$ and $\pi^{0,1} : \Omega^1_{\mathbb{C}}(X, E) \to \Omega^{0,1}(X, E)$ denote the usual projection maps.

Definition 1.3.5. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. Let ∇ be a connection on E. The *curvature* R^E is the $\mathrm{End}(E)$-valued 2-form defined by

$$R^E(\xi, \eta)\sigma := \nabla_\xi \nabla_\eta \sigma - \nabla_\eta \nabla_\xi \sigma - \nabla_{[\xi,\eta]}\sigma$$

where $\xi, \eta \in \Gamma(T_{\mathbb{C}}X)$ and $\sigma \in \Gamma(E)$.

Let $\{\partial_i\}$ denote a local coordinate frame on $T^{1,0}X$, $\{\phi_\alpha\}$ denote a smooth local frame on E. Then, we may express the coefficients of this form, denoted $R^{E\ \ \beta}_{i\bar{j}\alpha}$, defined by

$$R(\partial_i, \partial_{\bar{j}})\phi_\alpha = R^{E\ \ \beta}_{i\bar{j}\alpha}\phi_\beta.$$

Definition 1.3.6. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. A *Hermitian fibre metric* h on E is a family of Hermitian inner products $h_p : E_p \times E_p \to \mathbb{C}$ smoothly parameterised by $p \in X$. A holomorphic vector bundle endowed with a Hermitian fibre metric is referred to as a *Hermitian vector bundle.*

Endowing a holomorphic vector bundle with a Hermitian fibre metric allows us to consider R^E as a scalar-valued map, by setting

$$R^E(\xi, \eta, \sigma, \tau) := \langle R^E(\xi, \eta)\sigma, \tau\rangle$$

where $\xi, \eta \in \Gamma(T_{\mathbb{C}}X)$ and $\sigma, \tau \in \Gamma(E)$. We may therefore define the components of this map to be

$$R^E_{i\bar{j}\alpha\overline{\beta}} := R^E(\partial_i, \partial_{\bar{j}}, \phi_\alpha, \phi_\beta)$$

where the same notation is used as before. Lowering the index yields that $R^E_{i\bar{j}\alpha\overline{\beta}} = h_{\gamma\overline{\beta}} R^{E\ \ \gamma}_{i\bar{j}\alpha}$, where $h_{\gamma\overline{\beta}} := h(\phi_\gamma, \phi_\beta)$. We may also define the connection coefficients $\Gamma^\beta_{i\alpha}$ as $\nabla_{\partial_i}\phi_\alpha := \Gamma^\beta_{i\alpha}\phi_\beta$.

Definition 1.3.7. A connection on a Hermitian vector bundle is called *Hermitian* if it is compatible the Hermitian fibre metric h i.e., $\nabla h = 0$.

Proposition 1.3.8. Let X denote a complex manifold. If ∇ on $T^{1,0}X$ is Hermitian, then

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar{z}^j} = \overline{\nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^j}}.$$

Proof. Let $(x^1, \dots, x^n, y^1, \dots, y^n)$ denote smooth local coordinate son X, and write $\frac{\partial}{\partial z^i}$ as $\frac{\partial}{\partial x^i} - \sqrt{-1}J\frac{\partial}{\partial y^i}$. The following is a consequence of the linearity of ∇ and $\nabla_X(JY) = J\nabla_X Y$ for all $X, Y \in T_{\mathbb{C}}X$.

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar{z}^j} &= \nabla_{\partial_{x^i} - \sqrt{-1}J\partial_{y^i}}(\partial_{x^j} + \sqrt{-1}J\partial_{y^j}) \\
&= \nabla_{\partial_{x^i}}\partial_{x^j} - \sqrt{-1}\nabla_{J\partial_{y^i}}\partial_{x^j} + J(\nabla_{J\partial_{y^i}}\partial_{y^j}) + \sqrt{-1}J(\nabla_{\partial_{x^i}}\partial_{y^j})
\end{aligned}$$

We calculate $\nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^j}$ in a similar fashion.

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^j} &= \nabla_{\partial_{x^i} + \sqrt{-1}J\partial_{y^i}}(\partial_{x^j} - \sqrt{-1}J\partial_{y^j}) \\
&= \nabla_{\partial_{x^i}}\partial_{x^j} + \sqrt{-1}\nabla_{J\partial_{y^i}}\partial_{x^j} - \sqrt{-1}J\nabla_{\partial_{x^i}}\partial_{y^j} + J(\nabla_{J\partial_{y^i}}\partial_{y^j}).
\end{aligned}$$

We note that $\nabla_{\partial_{x^i}}\partial_{x^j}$ and $\nabla_{\partial_{y^i}}\partial_{y^j}$ are both real vector fields. Hence, conjugating yields the result. $\square$

Definition 1.3.9. Let X be a complex manifold. The *torsion* T of a Hermitian connection ∇ on $T_{\mathbb{C}}X$ is defined by

$$T(u,v) = \nabla_u v - \nabla_v u - [u,v], \qquad u,v \in T^{1,0}X$$

We may express the components of this tensor in the local frame $\{\partial_i\}$, which we define as

$$T_{ij}^k := \Gamma_{ij}^k - \Gamma_{ji}^k$$

We remind ourselves that imbuing a connection on the tangent bundle with metric compatibility and vanishing torsion yields a unique structure called the *Levi-Civita connection*. This connection is ill-fitting for complex manifolds however, as we shall see it rather restrictive on the complex structure (see Proposition 1.4.3). As we observe with Theorem 1.3.12, a connection useful for Hermitian geometry must be given different attributes. In order to construct this connection, we briefly discuss the Cartan theory of connections.

Definition 1.3.10. Let ∇ be a connection on a holomorphic vector bundle $E \to X$ over a complex manifold X. The *connection matrix* for ∇ (relative to a local frame $\{\phi_i\}$ for E) is the matrix $\omega = (\omega_j^i)$ of 1-forms given by

$$\nabla_X \phi_j = \sum_j \omega_j^i(X)\phi_j$$

for all $X \in TX$.

One may extend the definition of the connection ∇ to E-valued p-forms by forcing the Leibniz rule

$$\nabla(\omega \otimes \sigma) := d\omega \otimes \sigma + (-1)^p \omega \wedge \nabla\sigma$$

where $\omega \in \Omega^p(E)$ and $\sigma \in \Gamma(E)$.

Definition 1.3.11. Let ∇ be a connection on a holomorphic vector bundle $E \to X$ over a complex manifold X. The *curvature matrix* for ∇ (relative to a local frame $\{\phi_i\}$ for E) is the matrix $\Theta = (\Theta_j^i)$ of 2-forms given by

$$\nabla_X(\nabla_Y \phi_j) = \sum_i \Theta_j^i(X,Y)\phi_i$$

for all $X, Y \in TX$.

It is a straightforward computation to observe that the curvature and connection forms are related by the *Cartan structural equations*, which are given in a local coordinate frame by

$$\Theta_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k,$$

where Einstein summation notation is used.

Theorem 1.3.12 ([Bal06, p. 37])**.** Let E be a Hermitian vector bundle on a complex manifold X with Hermitian fibre metric h on E. Then there exists a unique compatible connection ∇ on E, such that $\nabla^{0,1} = \overline{\partial}$.

Proof. We first prove uniqueness. Assume that ∇ on E satisfies the conditions prescribed. Let ω_μ^ν be the connection forms for ω. First observe that ω is a (1,0)-form. Indeed, take a holomorphic frame $\{\phi_\alpha\}$, thus defining our connection forms as $\nabla\phi_\mu = \omega_\mu^\nu \phi_\nu$. Evaluating this explicitly, we obtain

$$\nabla\phi_\mu = \nabla^{1,0}\phi_\mu + \nabla^{0,1}\phi_\mu = \nabla^{1,0}\phi_\mu + \overline{\partial}\phi_\mu = \nabla^{1,0}\phi_\mu \in \Lambda^{1,0}(E).$$

We now consider $dh(\phi_\alpha, \phi_\beta)$. Let $h_{\alpha\overline{\beta}} = h(\phi_\alpha, \phi_\beta)$. Since $\nabla h = 0$, we have

$$\begin{aligned} dh(\phi_\alpha, \phi_\beta) &= h(\nabla \phi_\alpha, \phi_\beta) + h(\phi_\alpha, \nabla \phi_\beta) \\ &= h(\omega_\alpha^\gamma \phi_\gamma, \phi_\beta) + h(\phi_\alpha, \omega_\beta^\gamma \phi_\gamma) \\ &= \omega_\alpha^\gamma h_{\gamma\overline{\beta}} + \overline{\omega_\beta^\gamma h_{\gamma\overline{\alpha}}}. \end{aligned}$$

Since $dh(\phi_\alpha, \phi_\beta) = \partial h_{\alpha\overline{\beta}} + \overline{\partial} h_{\alpha\overline{\beta}}$ and ω is a (1,0)-form, we may compare both sides of the equation to obtain

$$\omega = h^{-1} \cdot \partial h$$

Since this expression only depends on h, we therefore have that ∇ is uniquely determined. Existence now follows from defining ω as per above, which is well-defined as it is independent of the choice of frame. We now show that it is hermitian and that $\nabla^{0,1} = \overline{\partial}$. Let $\sigma = \sigma^\alpha \phi_\alpha$ be a smooth section of E. Then,

$$\begin{aligned} \nabla(\sigma^\alpha \phi_\alpha) &= d\sigma^\alpha \otimes \phi_\alpha + \sigma^\alpha \nabla \phi_\alpha \\ &= (\partial \sigma^\alpha + \overline{\partial} \sigma^\alpha) \otimes \phi_\alpha + \sigma^\alpha \omega_\alpha^\gamma \phi_\gamma. \end{aligned}$$

Since ω is a $(1,0)$-form by construction, we have that $\nabla^{0,1} = \overline{\partial}$. We now prove metric compatability. Let $\{\partial_i\}$ denote the coordinate frame on $T_{\mathbb{C}}X$. Consider $h(\nabla_{\partial_k}\phi_\alpha, \phi_\beta) + h(\phi_\alpha, \nabla_{\partial_{\overline{k}}}\phi_\beta)$. Since ϕ_α is a holomorphic frame and $\nabla^{0,1} = \overline{\partial}$, we have that $\nabla_{\partial_{\overline{k}}}\phi_\beta = \overline{\partial}(\phi_\beta) = 0$. Hence,

$$h(\nabla_{\partial_k}\phi_\alpha, \phi_\beta) + h(\phi_\alpha, \nabla_{\partial_{\overline{k}}}\phi_\beta) = h(\Gamma_{k\alpha}^\gamma \phi_\gamma, \phi_\beta) = h_{\gamma\overline{\beta}}(h^{\gamma\overline{\delta}} \partial_k h_{\alpha\overline{\delta}}) = \partial_k h(\phi_\alpha, \phi_\beta).$$

where the Christoffel symbols are easily calculated via the relation $\omega = h^{-1} \cdot \partial h$. This implies that ∇ is hermitian. $\square$

Definition 1.3.13. The connection ∇ on E given by Theorem 1.3.12 is called the *Chern connection.*

The only non-trivial Christoffel symbols of the Chern connection are $\Gamma_{i\alpha}^\beta$ and $\Gamma_{\overline{i}\overline{\alpha}}^{\overline{\beta}}$.

Example 1.3.14. Let $(\mathcal{L}, h) \to X$ be a holomorphic line bundle over a complex manifold with Hermitian fibre metric h. Further, we let ∇ denote the corresponding Chern connection of $(\mathcal{L}, h) \to X$. The proof of the Chern connection demonstrates that the connection matrix ω is given by $h^{-1} \cdot \partial h = \partial \log(h)$. Hence, the curvature matrix $\Theta^{(\mathcal{L},h)}$ is given by

$$\Theta^{(\mathcal{L},h)} = d\omega + \omega \wedge \omega = d\omega = \overline{\partial}\partial \log(h) = -\partial\overline{\partial} \log(h).$$

Remark 1.3.15. Since a Hermitian metric on a line bundle $\mathcal{L} \to X$ is completely determined by a strictly positive function, it is customary to write $h = e^{-\varphi}$ for some smooth function φ on X. Computing the curvature explicitly, we obtain

$$\Theta^{(\mathcal{L},h)} = \partial\overline{\partial}\varphi.$$

Hence, $\Theta^{(\mathcal{L},h)}$ is a positive $(1,1)$-form if φ is plurisubharmonic. Any such line bundle where Θ is positive is referred to as a *positive line bundle.*

We now remind ourselves of a particularly important class of covariant tensor fields.

Reminder 1.3.16 (Riemannian metrics). Let M be a smooth manifold. A *Riemannian metric* g on M is a smooth section of $T^{(0,2)}(M)$ which is symmetric and positive definite. A smooth manifold equipped with a Riemannian metric is referred to as a *Riemannian manifold.*

If X is a complex manifold, it is natural to impose the following condition.

Definition 1.3.17. Let (X, J) be a complex manifold. A Riemannian metric g on X is called *Hermitian* if

$$g(Ju, Jv) = g(u, v)$$

for all $u, v \in TX$. The pair (X, g) is referred to as a *Hermitian manifold.* The *associated (1,1)-form* of a Hermitian metric g is defined by

$$\omega(u, v) := g(Ju, v).$$

Proposition 1.3.18 ([Bro22c, p. 78])**.** All complex manifolds admit a Hermitian metric.

Proof. We leverage the fact that all manifolds admit a Riemannian metric g and define $\hat{g}$ by

$$\hat{g}(u, v) := g(u, v) + g(Ju, Jv).$$

This is a Hermitian metric by construction. ☐

Given a Hermitian metric and complex structure, a unique Hermitian fibre metric may be constructed on the holomorphic tangent bundle.

Proposition 1.3.19 ([Bro22c, p. 79])**.** Let (X, g) be a Hermitian manifold. Then, we may define a Hermitian fibre metric $h : T^{1,0}X \times T^{1,0}X \to \mathbb{C}$ by $\mathrm{Re}(h) = g, \mathrm{Im}(h) = -\omega$.

Proof. Let $\{x^1, \dots x^n, x^{n+1}, \dots x^{2n}\}$ denote smooth (real-valued) local coordinates on X. Introduce the notation $I := i+n$ to apply the Einstein summation convention. Assume the complex structure acts according to

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^I}, \qquad J\left(\frac{\partial}{\partial x^I}\right) = -\frac{\partial}{\partial x^i}.$$

In these coordinates, the Riemannian metric reads

$$g = g_{ik} dx^i \otimes dx^k + g_{iK} dx^i \otimes dx^K + g_{Ik} dx^I \otimes dx^k + g_{IK} dx^I \otimes dx^K$$

From the compatibility condition $g(u, v) = g(Ju, Jv)$, we have

$$g_{ik} = g_{IK}, \qquad g_{iK} = g_{Ki} = -g_{kI} = -g_{Ik}.$$

Introduce the following complex coordinates $\{z^i, \overline{z}^i\}$ on X, given by

$$z^i = x^i + \sqrt{-1}x^I, \qquad \overline{z}^i = x^i - \sqrt{-1}x^I.$$

Then $dz^i = dx^i + \sqrt{-1}dx^I$ and $dz^i = dx^i - \sqrt{-1}dx^I$, and hence,

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial x^I}\right), \qquad \frac{\partial}{\partial \overline{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial x^I}\right)$$

We note that

$$h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = h\left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^j}\right) = 0.$$

Moreover,

$$h_{i\overline{j}} := h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\right) = g_{ij} - \sqrt{-1}g_{Ji}.$$

This implies that h is non-degenerate on $T^{1,0}X$, meaning that it defines a Hermitian fibre metric. That $g_{Ji} = g\left(J\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ proves the proposition. ☐

A Hermitian fibre metric h can be written in local holomorphic coordinates (z^i) by

$$h = h_{i\bar{j}} dz^i \otimes d\bar{z}^j$$

where Einstein notation is used. Hence, in local coordinates, the associated $(1,1)$-form is written as

$$\omega = \frac{\sqrt{-1}}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Observe that

$$\overline{\omega} = -\frac{\sqrt{-1}}{2} h_{j\bar{i}} d\bar{z}^i \wedge dz^j = \frac{\sqrt{-1}}{2} h_{j\bar{i}} dz^j \wedge d\bar{z}^i = \omega,$$

implying that ω is a real-valued form. This observation leads some authors to describe ω the *associated 2-form* of g.

Remark 1.3.20. Above, we see three separate structures: the Hermitian metric g, the Hermitian fibre metric h, and the associated $(1,1)$-form ω. These structures often have different names depending on the author. As an example to illustrate this inconsistency in nomenclature, all such structures are - frustratingly - referred to as the Hermitian metric. Although ω is in fact a $(1,1)$-form, it is sometimes referred to in this way as a tradition. To compress notation, ω_g denotes the $(1,1)$-form ω with underlying compatible metric g. Throughout this project, we sometimes refer to ω_g as the Hermitian metric, as many examples of such metrics are furnished by considering $(1,1)$-forms (see Example 1.3.25, Example 1.3.26).

This proposition is a useful characterisation of Hermitian metrics that unearths important results.

Example 1.3.21. Consider $\mathbb{C}^n$ with the metric $h = \sum_{k=1}^n dz^k \otimes d\bar{z}^k$. In light of Proposition 1.3.19, we apply the change of coordinates $z^i = x^i + \sqrt{-1}y^i$. Therefore, our metric now reads

$$h = \sum_{k=1}^n (dx^k \otimes dx^k + dy^k \otimes dy^k) - \sqrt{-1} \sum_{k=1}^n (dx^k \otimes dy^k - dy^k \otimes dx^k)$$

We observe that the underlying Riemannian metric is merely the Euclidean metric on $\mathbb{R}^{2n}$. The Euclidean metric is invariant under J (that is to say J is a linear isometry), meaning that h is Hermitian.

Definition 1.3.22. Let (X, g) be a Hermitian manifold. The Hermitian metric g is said to be a *Kähler metric* if the associated $(1,1)$-form is closed

$$d\omega = 0.$$

A Hermitian manifold (X, g, ω) is said to be *Kähler* if g is a Kähler metric. A complex manifold X is said to be *Kähler* if it supports a Kähler metric.

Example 1.3.23. A simple example of a Kähler manifold is $\mathbb{C}^n$ equipped with the *flat metric* $\delta = \sum_{i=1}^n dz^i \otimes d\bar{z}^i$. The associated (1,1)-form is $\omega = \sqrt{-1} \sum_{i=1}^n dz^i \wedge d\bar{z}^i$, which is obviously closed.

Example 1.3.24. The flat metric on $\mathbb{C}^n$ descends to a Kähler metric on $\mathbb{C}^n/\Lambda$, the complex n-torus with lattice $\Lambda \subseteq \mathbb{C}^n$, still referred to as the *flat metric*.

Example 1.3.25. Let $\mathbb{P}^n$ denote the complex projective space. We define the *Fubini–Study metric* as follows: Let $\{U_i\}$ denote the usual covering for $\mathbb{P}^n$ as defined in Example 1.1.25. Consider the following form defined locally on U_i.

$$\omega_i := \sqrt{-1}\partial\bar{\partial} \log \left(\sum_{\ell=0}^n \left| \frac{z^\ell}{z^i} \right|^2 \right) \in \Omega^{1,1}(U_i).$$

Under the coordinate chart map φ_i, the expression reads

$$\omega_i := \sqrt{-1}\partial\overline{\partial}\log\left(1+\sum_{\ell=0}^{n}|w^k|^2\right).$$

To show that ω is globally defined, it suffices to prove that $\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}$. Observe,

$$\log\left(\sum_{\ell=0}^{n}\left|\frac{z^\ell}{z^i}\right|^2\right) = \log\left(\left|\frac{z^j}{z^i}\right|\sum_{\ell=0}^{n}\left|\frac{z^\ell}{z^j}\right|^2\right) = \log\left(\left|\frac{z^j}{z^i}\right|\right) + \log\left(\sum_{\ell=0}^{n}\left|\frac{z^\ell}{z^j}\right|^2\right)$$

It therefore suffices to show that $\partial\overline{\partial}\log(|z^j/z^i|) = 0$. In local holomorphic coordinates (i.e., under φ_i), we have

$$\partial\overline{\partial}\log(z^j\overline{z}^j) = \partial\left(\sum_{k=1}^{n}\frac{z^j\delta_k^j}{z^j\overline{z}^j}d\overline{z}^k\right) = \partial\left(\frac{1}{\overline{z}^j}d\overline{z}^j\right) = 0$$

Hence, we have that $\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}$. Denote the global extension of this $(1,1)$-form as ω_{FS}. That ω_{FS} is closed and a real 2-form follows from Proposition 1.2.17, as $\partial\overline{\partial} = -\overline{\partial}\partial$ and $\partial^2 = \overline{\partial}^2 = 0$. To show it is positive definite, we calculate ω_{FS} explicitly:

$$\sqrt{-1}\partial\overline{\partial}\log\left(1+\sum_{i=1}^{n}|w^i|^2\right) = \sqrt{-1}\sum_{i,j=1}^{n}\frac{(1+\sum|w^i|^2)\delta_j^i - \overline{w}^i w^j}{(1+\sum|w^i|^2)^2}dw^i\wedge d\overline{w}^j.$$

It is straight forward to check that ω_{FS} is $\mathrm{U}(n+1)$ invariant. Since $\mathrm{U}(n+1)$ acts transitively on $\mathbb{P}^n$, it suffices to check wether ω_{FS} is positive definite at one point. At the origin, we have

$$\omega_{FS} = \sqrt{-1}\sum_{i,j=1}^{n}\delta_j^i dz^i\wedge d\overline{z}^j.$$

The coefficients correspond to the identity matrix which is positive definite. Hence, the metric associated with ω_{FS} is indeed a Kähler metric.

Example 1.3.26. Recall that for a domain $\Omega \subseteq \mathbb{C}^n$, one may define the associated Bergman kernel $K(z,\overline{\zeta})$. For bounded domains, $K(z,\overline{z})$ is strictly plurisubharmonic, meaning we can define a Hermitian metric on Ω. This metric is referred to as the *Bergman metric*, and is given by the 2-form

$$\omega_\Omega := \sqrt{-1}\partial\overline{\partial}\log K(z,\overline{z}).$$

The Bergman metric is a Kähler metric. We now compute the Bergman metric on the ball.

$$\begin{aligned}\omega_{\mathbb{B}^n} &= \sqrt{-1}\partial\overline{\partial}\log\left(\frac{n!}{\pi^n\log(1-|z|^2)^{n+1}}\right)\\ &= -(n+1)\sqrt{-1}\partial\overline{\partial}\log(1-|z|^2).\end{aligned}$$

Computing this directly is routine. Hence, the Bergman metric on the ball $g_{\mathbb{B}^n}$ is given by

$$g_{\mathbb{B}^n} = (n+1)\frac{(1-|z|)^2\delta_j^i + \overline{z}^i z^j}{(1-|z|^2)^2}dz^i\otimes d\overline{z}^j.$$

Observe that for $n=1$, the Bergman metric coincides with the classical Poincaré metric on the unit disk

$$g_{\mathbb{B}} = \frac{2}{(1-|z|^2)^2}dz^i\otimes d\overline{z}^j.$$

Example 1.3.27. Let (Y,h) be a Hermitian manifold, and $f: X\to Y$ be a holomorphic map. Consider the pullback of h, denoted f^*h. If f^*h is positive definite, then it is referred to as the *pullback metric* on X.

Proposition 1.3.28. Let $f : X \to Y$ be a holomorphic map, and h denote a Hermitian metric on Y. Then, f is a holomorphic immersion if and only if f^*h is a Hermitian metric.

This proposition furnishes an important subclass of pullback metrics. Since the inclusion map $\iota : S \hookrightarrow (Y, h)$ of a submanifold S is a holomorphic immersion, we have that $\iota^* h$ is a Hermitian metric by the above. Such a metric is referred to as the *induced metric*. We may also define an equivalence of geometric structures.

Definition 1.3.29. Let (X, g) and $(\widetilde{X}, \widetilde{g})$ be Hermitian manifolds. An *isometry* is a diffeomorphism $f : M \to \widetilde{M}$ such that $f^*\widetilde{g} = g$. In this case, we say that (M, g) and $(\widetilde{M}, \widetilde{g})$ are *isometric*.

It may also be shown that the set of isometries on a Hermitian manifold (X, g) form a group under composition, which we denote $\mathrm{Iso}(M, g)$. This group is referred to as the *isometry group*. We now state more examples of Hermitian and Kähler metrics.

Example 1.3.30. The Kähler condition is preserved under holomorphic immersions. In particular, let $f : X \to (Y, \eta, h)$ be a holomorphic immersion between complex manifolds. The pullback of η by f is given by $(f^*\eta)(u, v) = \eta(df(u), df(v))$. Since f is holomorphic, the complex structure is preserved and $f^*\eta$ is non-degenerate as f is an immersion.

Example 1.3.31. The previous example implies that any complex submanifold of a Kähler manifold is also a Kähler manifold. The submanifold inherits the metric on the ambient manifold. Hence, all projective and Stein manifolds are Kähler, as they inherit the Fubini–Study metric and flat metric respectively.

Example 1.3.32. Consider a Hermitian manifold X such that $\dim_{\mathbb{C}}(X) = 1$. As a corollary of Proposition 1.3.18 and Proposition 1.3.19, all complex manifolds admit a Hermitian metric h (of the form $h dz \otimes d\overline{z}$ where $h : X \to \mathbb{C}$ is smooth) with associated $(1,1)$-form ω. Calculating $d\omega$, we observe,

$$d\omega = \left(\frac{\partial h}{\partial z} dz + \frac{\partial h}{\partial \overline{z}} d\overline{z} \right) \wedge dz \wedge d\overline{z} = \frac{\partial h}{\partial z} dz \wedge dz \wedge d\overline{z} - \frac{\partial h}{\partial \overline{z}} d\overline{z} \wedge d\overline{z} \wedge dz = 0.$$

Therefore, all Riemann surfaces are Kähler.

The following theorem demonstrates that the Kähler condition is severely restricted by the underlying topology of the manifold.

Theorem 1.3.33 ([Mor07, p. 109])**.** Let $k \in \mathbb{Z}_{\geq 0}$ and X be a compact Kähler manifold. Then, $b_{2k+1}(X)$ is even.

The above theorem then constructs one of the first examples of a complex manifold which admits no Kähler metrics.

Non-Example 1.3.34. The primary Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$ is a compact Hermitian manifold that admits no Kähler metric. Indeed, the Künneth formula yields that $b_1(\mathbb{S}^3 \times \mathbb{S}^1) = 1$, which is not even.

There are various classes of Hermitian metrics that relax the Kähler condition but still retain enough structure for meaningful study.

Definition 1.3.35. Let (X, g) be a Hermitian manifold of (complex) dimension n, where g has associated $(1,1)$-form ω. We say that g is

(i) *pluriclosed* if $\partial\overline{\partial}\omega = 0$.
(ii) *balanced* if $d\omega^{n-1} = 0$,

where $\omega^{n-1} := \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$.

Example 1.3.36. Although the primary Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$ admits no Kähler metric, the metric described by the (1,1)-form

$$\omega_0 := \sqrt{-1}\frac{\delta^i_j}{|z|^2} dz^i \wedge d\overline{z}^j$$

is a Hermitian metric which is pluriclosed. The metric associated with this form, denoted g_0, is referred to as the *Boothby metric*. We show that g_0 is pluriclosed by direct computation, removing the factor of $\sqrt{-1}$ for simplicity.

$$\begin{aligned}
\overline{\partial}\omega_0 &= \frac{-\delta^i_j z^\ell}{|z|^4} dz^i \wedge d\overline{z}^j \wedge d\overline{z}^\ell \\
\partial\overline{\partial}\omega_0 &= \frac{-\delta^k_\ell \delta^i_j |z|^2 + 2\delta^k_\ell \overline{z}^i z^j}{|z|^6} dz^i \wedge d\overline{z}^j \wedge dz^k \wedge d\overline{z}^\ell
\end{aligned}$$

Summing over i, j, k, ℓ and cancelling out several terms via the antisymmetry of the wedge operator, we obtain

$$\begin{aligned}
\partial\overline{\partial}\omega_0 &= \frac{-2}{|z|^4}(dz^1 \wedge d\overline{z}^1 \wedge dz^2 \wedge d\overline{z}^2) + \frac{2|z^1|^2}{|z|^6}(dz^1 \wedge d\overline{z}^1 \wedge dz^2 \wedge d\overline{z}^2) + \frac{2|z^2|^2}{|z^2|^6}(dz^1 \wedge d\overline{z}^1 \wedge dz^2 \wedge d\overline{z}^2) \\
&= \frac{2(|z^1|^2 + |z^2|^2 - |z|^2)}{|z|^6}(dz^1 \wedge d\overline{z}^1 \wedge dz^2 \wedge d\overline{z}^2) \\
&= 0.
\end{aligned}$$

Example 1.3.37. Let $\mathrm{GL}(3, \mathbb{C})$ denote invertible 3×3 matrices with entries in $\mathbb{C}$. We consider the subgroup G, defined as

$$G := \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} : z^1, z^2, z^3 \in \mathbb{C} \right\}.$$

This subgroup is a complex Lie group biholomorphic to $\mathbb{C}^3$, where multiplication is given by

$$(z^1, z^2, z^3) \cdot (w^1, w^2, w^3) = (z^1 + w^1, z^2 + w^2, z^3 + z^1 w^2 + z^3).$$

An *Iwasawa manifold* is the left coset space G/Γ, where Γ is a discrete subgroup of G such that G/Γ is compact. A simple example is the *standard Iwasawa manifold*, where matrix entries in Γ are given by the Gaussian integers i.e., $n + \sqrt{-1}m$ for $n, m \in \mathbb{Z}$. It was proven by Michelsohn [Mic82] that although the standard Iwasawa manifold admits no Kähler metrics, it does admit a balanced metric.

Proposition 1.3.38 ([Zhe00, p. 173])**.** Let (X, g) be a Kähler manifold. Then in local coordinates $(z^1, ..., z^n)$, the metric $g = \sum_{i,j=1}^n g_{i\overline{j}} dz^i \otimes d\overline{z}^j$ has the following symmetry:

$$\frac{\partial g_{i\overline{j}}}{\partial z^k} = \frac{\partial g_{k\overline{j}}}{\partial z^i}, \qquad \frac{\partial g_{i\overline{j}}}{\partial \overline{z}^k} = \frac{\partial g_{i\overline{k}}}{\partial \overline{z}^j}.$$

Proof. The associated $(1,1)$-form of the metric g is given as $\omega = \sqrt{-1}\sum_{i,j=1}^n g_{i\overline{j}} dz^i \wedge d\overline{z}^j$. Then,

$$0 = d\omega = \partial\omega + \overline{\partial}\omega$$

$$= \sqrt{-1}\sum_{i,j=1}^{n}\partial(g_{i\bar{j}})\wedge dz^i\wedge d\overline{z}^j + \sqrt{-1}\sum_{i,j=1}^{n}\overline{\partial}(g_{i\bar{j}})\wedge dz^i\wedge d\overline{z}^j$$
$$= \sqrt{-1}\sum_{i,j,k=1}^{n}\frac{\partial g_{i\bar{j}}}{\partial z^k}dz^k\wedge dz^i\wedge d\overline{z}^j + \sqrt{-1}\sum_{i,j,k=1}^{n}\frac{\partial g_{i\bar{j}}}{\partial \overline{z}^k}d\overline{z}^k\wedge dz^i\wedge d\overline{z}^j$$
$$= \sqrt{-1}\sum_{j=1}^{n}\left(\sum_{i<k}\frac{\partial g_{i\bar{j}}}{\partial z^k}dz^k\wedge dz^i\wedge d\overline{z}^j + \sum_{i>k}\frac{\partial g_{i\bar{j}}}{\partial z^k}dz^k\wedge dz^i\wedge d\overline{z}^j\right)$$
$$+ \sqrt{-1}\sum_{i=1}^{n}\left(\sum_{j<k}\frac{\partial g_{i\bar{j}}}{\partial \overline{z}^k}d\overline{z}^k\wedge dz^i\wedge d\overline{z}^j + \sum_{j>k}\frac{\partial g_{i\bar{j}}}{\partial \overline{z}^k}d\overline{z}^k\wedge dz^i\wedge d\overline{z}^j\right)$$
$$= \sqrt{-1}\sum_{i,j,k=1}^{n}\left(\frac{\partial g_{i\bar{j}}}{\partial z^k} - \frac{\partial g_{k\bar{j}}}{\partial z^i}\right)dz^i\wedge dz^k\wedge d\overline{z}^j + \sqrt{-1}\sum_{i,j,k=1}^{n}\left(\frac{\partial g_{i\bar{j}}}{\partial \overline{z}^k} - \frac{\partial g_{i\bar{k}}}{\partial \overline{z}^j}\right)d\overline{z}^k\wedge dz^i\wedge d\overline{z}^j$$

As the first and second terms lie in different vector bundles ($\Omega^{2,1}(X)$ and $\Omega^{1,2}(X)$ respectively), our symmetries are obtained. $\square$

1.4. **Curvature of Hermitian Metrics.** As a Hermitian metric is also a Hermitian fibre metric on the holomorphic tangent bundle, we may consider its Chern connection. Although we define this connection on $T^{1,0}X$, we may extend it to $T^{0,1}X$ by conjugation (see Proposition 1.3.8 for details). The extension of the Chern connection to $T_{\mathbb{C}}X$ is still referred to as the Chern connection.

Proposition 1.4.1. Let (X, g) denote a Hermitian manifold. Then, the Chern connection in a local coordinate frame $\frac{\partial}{\partial z^i}$ is given by

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z^i\partial\overline{z}_j} + g^{p\bar{q}}\frac{\partial g_{k\bar{q}}}{\partial z^i}\frac{\partial g_{p\bar{\ell}}}{\partial \overline{z}_j}.$$

Proof. We compute $R_{i\bar{j}k}{}^p$. Theorem 1.3.12 yields that $\nabla_{\partial_{\bar{k}}}s = \overline{\partial}_k s$ for a smooth section s of a vector bundle E. Further, if s is holomorphic, $\nabla_{\partial_{\bar{k}}}s = 0$. Hence,

$$R_{i\bar{j}k}{}^p\partial_p = (\nabla_{\partial_i}\nabla_{\partial_{\bar{j}}} - \nabla_{\partial_{\bar{j}}}\nabla_{\partial_i})\partial_k = -\nabla_{\partial_{\bar{j}}}(\Gamma^p_{ik}\partial_p) = -(\partial_{\bar{j}}\Gamma^p_{ik})\partial_p$$

From Theorem 1.3.12, we have that $R_{i\bar{j}k}{}^p = -\partial_{\bar{j}}(g^{p\bar{q}}\partial_i g_{k\bar{q}})$. We now consider $\partial_{\bar{j}}g^{p\bar{q}}$. Consider that $g^{p\overline{m}}g_{p\bar{q}} = \delta^m_q$. Applying $\partial_{\bar{j}}$ to both sides, we obtain the following.

$$\partial_{\bar{j}}(g^{p\overline{m}}g_{p\bar{q}}) = 0$$
$$(\partial_{\bar{j}}g^{p\overline{m}})g_{p\bar{q}} + g^{p\overline{m}}(\partial_{\bar{j}}g_{p\bar{q}}) = 0$$

Rearranging for the desired term, we then obtain,

$$\partial_{\bar{j}}g^{a\overline{m}} = -g^{a\bar{q}}g^{p\overline{m}}(\partial_{\bar{j}}g_{p\bar{q}}).$$

Using this, observe,

$$R_{i\bar{j}k}{}^p = g^{p\overline{m}}g^{n\bar{q}}(\partial_{\bar{j}}g_{n\overline{m}})(\partial_i g_{k\bar{q}}) - g^{p\bar{q}}\partial_i\partial_{\bar{j}}g_{k\bar{q}}.$$

Recalling that $R_{i\bar{j}k\bar{\ell}} = R_{i\bar{j}k}{}^p g_{p\bar{\ell}}$, we conclude that

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z^i\partial\overline{z}_j} + g^{p\bar{q}}\frac{\partial g_{k\bar{q}}}{\partial z^i}\frac{\partial g_{p\bar{\ell}}}{\partial \overline{z}_j}.$$

$\square$

It is an immediate consequence of the above that $\overline{R_{i\bar{j}k\bar{\ell}}} = R_{j\bar{i}\ell\bar{k}}$.

Reminder 1.4.2. The Levi–Civita connection. Let M denote a smooth manifold. The *torsion* T of a connection ∇ on TM, defined as the $(1,2)$-tensor field $T(u,v) = \nabla_u v - \nabla_v u - [u,v]$, where $[\cdot,\cdot]$ denotes the Lie bracket. We say that a connection is *torsion-free* if T vanishes identically. One of the landmark results of Riemannian geometry is that there exists a unique connection which is both compatible (i.e., $\nabla g = 0$) and torsion-free on any Riemannian manifold (M,g) [Lee18]. This connection to referred to as the *Levi–Civita connection.*

As we now observe, the Levi–Civita connection is ill-fitting for studying Hermitian (not necessarily Kähler) manifolds.

Proposition 1.4.3 ([Zhe00, p. 173])**.** On a Hermitian manifold (X,g), the Chern connection coincides with the Levi-Civita connection if and only if (X,g) is Kähler.

Proof. If $\nabla^c = \nabla^{LC}$, we observe that ${}^cT = 0$. Computing the torsion coefficients of the Chern connection, we have

$$ {}^cT^k_{ij} = g^{k\bar{\ell}}(\partial_i g_{j\bar{\ell}} - \partial_j g_{i\bar{\ell}}) $$

This implies that $\partial_i g_{j\bar{\ell}} = \partial_j g_{i\bar{\ell}}$, meaning that the Kähler condition is met. For the other direction, assume that (X,g) is Kähler. Then, the symmetries of the Kähler metric as proven in Proposition 1.3.38 yield that ${}^cT = 0$. Since we have that $\nabla^c g = 0$ and ${}^cT = 0$, the uniqueness of the Levi-Civita connection implies that $\nabla^c = \nabla^{LC}$. □

The above demonstrates that the Chern connection is a much more natural object than the Levi–Civita connection in the realm of Hermitian geometry. With the Chern connection, we now construct the associated notions of curvature in the Hermitian category.

Definition 1.4.4. The trace of the curvature of the Chern connection yields four *Chern Ricci curvatures*, given by $\mathrm{Ric}^{(k)} := \mathrm{Ric}^{(k)}_{i\bar{j}} dz^i \otimes d\bar{z}^j$, where

$$ \mathrm{Ric}^{(1)}_{i\bar{j}} := g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}, \qquad \mathrm{Ric}^{(2)}_{k\bar{\ell}} := g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}, \qquad \mathrm{Ric}^{(3)}_{k\bar{\ell}} := g^{i\bar{\ell}} R_{i\bar{j}k\bar{\ell}}, \qquad \mathrm{Ric}^{(4)}_{i\bar{\ell}} := g^{k\bar{j}} R_{i\bar{j}k\bar{\ell}}. $$

Note that $\mathrm{Ric}^{(4)}_{i\bar{\ell}} := g^{k\bar{j}} R_{i\bar{j}k\bar{\ell}}$ is the conjugate of $\mathrm{Ric}^{(3)}_{k\bar{\ell}}$, inherited from the symmetry $\overline{R_{i\bar{j}k\bar{\ell}}} = R_{j\bar{i}\ell\bar{k}}$.

Remark 1.4.5. The Chern Ricci curvatures are relatively new in the landscape of complex geometry. Recent developments have focused on the first Chern Ricci curvature. Particularly, compact non-Kähler manifolds with $\mathrm{Ric}^{(1)} = 0$, referred to as *non-Kähler Calabi–Yau* manifolds, have been of keen interest. Such manifolds were systematically studied by Tosatti [Tos15] through reducing the equation $\mathrm{Ric}^{(1)} = 0$ to fully non-linear second order elliptic partial differential equation, called a complex *Monge–Ampére* equation. We shall observe the Kähler case later in this thesis.

We now show that these Chern Ricci curvatures coincide if g is Kähler.

Proposition 1.4.6. Let g be a Kähler metric. Then, the Chern curvature tensor R has the following symmetry:

$$ R_{i\bar{j}k\bar{\ell}} = R_{i\bar{\ell}k\bar{j}} = R_{k\bar{j}i\bar{\ell}} = R_{k\bar{\ell}i\bar{j}}. $$

Proof. Since we are using the Levi–Civita connection, we may leverage its symmetries to obtain $R_{i\bar{j}k\bar{\ell}} = R_{k\bar{\ell}i\bar{j}}$ and $R_{i\bar{\ell}k\bar{j}} = R_{k\bar{j}i\bar{\ell}}$. Further, we obtain the following via the Bianchi identities of the Levi–Civita connection:

$$R_{i\bar{j}k\bar{\ell}} + R_{\bar{j}ki\bar{\ell}} + R_{ki\bar{j}\bar{\ell}} = 0.$$

As $R_{ki\bar{j}\bar{\ell}} = 0$, we obtain $R_{i\bar{j}k\bar{\ell}} = -R_{\bar{j}ki\bar{\ell}}$. Using the Bianchi identities again, we obtain,

$$R_{i\bar{j}k\bar{\ell}} = -R_{\bar{j}ki\bar{\ell}} = R_{k\bar{j}i\bar{\ell}} = R_{i\bar{\ell}k\bar{j}}.$$

□

Corollary 1.4.7. If g is Kähler, then all Chern Ricci curvatures coincide.

The novetly of Kähler and Hermitian metrics in complex geometry prompts the discussion of new types of curvature.

Definition 1.4.8. Let (X, g) denote a Hermitian manifold, and R denote the Chern curvature tensor of g. The *holomorphic bisectional curvature* of g in the direction of vectors $u, v \in T^{1,0}X$ is given by

$$\mathrm{HBC}_g(u, v) := \frac{1}{|u|_g^2 |v|_g^2} \sum_{i,\bar{j},k,\bar{\ell}} R_{i\bar{j}k\bar{\ell}} u^i \overline{u}^j v^k \overline{v}^\ell.$$

The holomorphic bisectional curvature earns its title through its relationship with the classical *sectional curvature*, defined as

$$\mathrm{Sec}_g(u, v) := \frac{R(u, v, v, u)}{|u|_g^2 |v|_g^2 - \langle u, v\rangle},$$

where $u, v \in TX$, R is the Riemannian curvature tensor, and $\langle \cdot, \cdot \rangle$ is alternative notation for the Hermitian metric g. For a Kähler metric, HBC_g splits into two sectional curvature terms:

$$\mathrm{HBC}_g(u, v) = \mathrm{Sec}_g(\mathrm{Re}u, \mathrm{Im}v) + \mathrm{Sec}_g(\mathrm{Re}v, \mathrm{Im}u).$$

The holomorphic bisectional curvature is a rather strong curvature term. For example, the Mori [Mor79], Siu–Yau [SY80] resolution of the Frankel conjecture demonstrates if (X, g) is a compact Kähler manifold with a Kähler metric of $\mathrm{HBC}_g > 0$, then X must be biholomorphic to n-dimensional projective space $\mathbb{P}^n$. We now introduce a weaker curvature term.

Definition 1.4.9. Let (X, g) denote a Hermitian manifold, and R denote the Chern curvature tensor of g. The *holomorphic sectional curvature* of g in the direction of a vector $u \in T^{1,0}X$ is given by

$$\mathrm{HSC}_g(u) := \frac{1}{|u|_g^4} \sum_{i,\bar{j},k,\bar{\ell}} R_{i\bar{j}k\bar{\ell}} u^i \overline{u}^j u^k \overline{u}^\ell.$$

For Kähler metrics, the holomorphic sectional curvature also shares a relationship with the classical sectional curvature. Indeed,

$$\mathrm{HSC}_g(v) = \mathrm{Sec}(\mathrm{Re}v, \mathrm{Im}v).$$

Remark 1.4.10. The relationship between the holomorphic sectional curvature and the Ricci curvature(s) is a subject of keen interest. The theorem of Wu–Yau [WY16] asserts that for a compact Kähler manifold (X, g) with $\mathrm{HSC}_g < 0$, there exists a Kähler metric $\widetilde{g}$ on X such that $\mathrm{Ric}_{\widetilde{g}} < 0$. Further, there has been recent evidence [Bro23] to support a positive analogue for the Wu–Yau theorem i.e.,

$$\mathrm{HSC}_g > 0 \implies \exists\, \widetilde{g} \text{ Kähler such that } \mathrm{Ric}_{\widetilde{g}} > 0.$$

There are, however, a number of examples which demonstrate that the link between HSC and Ric is not exact. For example, Hitchin [Hit75] demonstrated that Hirzebruch surfaces admit a certain subclass of Kähler metrics with HSC > 0, but do not admit any Kähler metric with Ric > 0. On the other hand, Wu–Yau [WY16] showed that for Fermat hypersurfaces of degree $d \geq n+2$ in $\mathbb{P}^n$ admit Kähler metrics with Ric < 0, but no Hermitian metrics with HSC < 0.

We now consider the curvature aspects of several Hermitian manifolds. Several we have already observed.

Proposition 1.4.11 ([Lee24, p. 256])**.** A Kähler manifold (X, g) is a space of constant holomorphic sectional curvature c if and only if

$$R_{i\bar{j}k\bar{\ell}} = \frac{1}{2}c(g_{i\bar{j}}g_{k\bar{\ell}} + g_{k\bar{j}}g_{i\bar{\ell}}).$$

The above proposition may immediately be leveraged for previously seen Kähler metrics.

Example 1.4.12. The Fubini–Study metric g_{FS} on complex projective space $\mathbb{P}^n$ does not have constant sectional curvature. In particular, the sectional curvatures lie between 1 and 4. However, it does have constant positive holomorphic sectional curvature. Since $\mathbb{P}^n$ is homogeneous, it suffices to calculate the holomorphic sectional curvature at a single point. Let $p = [1 : 0 : \cdots : 0]$. In coordinates, this corresponds to the origin. Then,

$$\begin{aligned} \partial_i g_{k\bar{\ell}} &= \frac{1}{2}\left(-\frac{\delta_{k\ell}\bar{z}^i}{(1+|z|^2)^2} - \frac{\delta_{i\ell}\bar{z}^k}{(1+|z|^2)^2} + \frac{2\bar{z}^k z^\ell \bar{z}^i}{(1+|z|^2)^3}\right) \\ \partial_{\bar{j}}\partial_i g_{k\bar{\ell}} &= \frac{-1}{2}(\delta_{k\ell}\delta_{ij} + \delta_{i\ell}\delta_{kj}). \end{aligned}$$

This means that at p, we have $g_{k\bar{\ell}} = \frac{1}{2}\delta_{k\ell}, \partial_i g_{k\bar{\ell}} = \partial_{\bar{i}} g_{k\bar{\ell}} = 0$. Further,

$$R_{i\bar{j}k\bar{\ell}} = \frac{1}{2}(\delta_{k\ell}\delta_{ij} + \delta_{i\ell}\delta_{kj}) = 2(g_{k\bar{\ell}}g_{i\bar{j}} + g_{i\bar{\ell}}g_{k\bar{j}}).$$

Applying Proposition 1.4.11 yields that HSC $\equiv 4$ on $(\mathbb{P}^n, g_{FS})$.

Example 1.4.13. The Euclidean metric $\bar{g}$ on $\mathbb{C}^n$ has constant holomorphic sectional curvature equal to 0. This does not require the lemma and can be clearly seen by computing $R_{i\bar{j}k\bar{\ell}}$.

Example 1.4.14. The Bergman metric $g_{\mathbb{B}}$ on the ball $\mathbb{B}^n$ has constant holomorphic sectional curvature. Following a similar process as in Example 1.4.12 yields that HSC $\equiv -4$.

The class of Hermitian metrics where HSC < 0 and HSC ≤ 0 will be of key importance towards the end of the project. For the former condition, we refer to such metrics as having *negative* HSC, while the latter we refer to as having *non-positive* HSC. We now provide examples of both.

Example 1.4.15. Let Ω denote a complex manifold biholomorphic to a bounded domain in $\mathbb{C}^n$. Further, we assume that for every point $p \in \Omega$, there is an is a holomorphic involution $\varphi_p : \Omega \to \Omega$ with p as a unique fixed point. That is, $\varphi_p(p) = p$ and $d\varphi_p = -\mathrm{Id}$ for all $p \in \Omega$, where φ is an isometry. Such a manifold is referred to as a *bounded symmetric domain*. Endowing Ω with the Bergman metric, we obtain HSC < 0.

Example 1.4.16. Let $\mathbb{B}^n$ denote the ball in $\mathbb{C}^n$. As stated prior, imbuing the ball with the Bergman metric yields a Kähler manifold of negative sectional curvature. This construction also holds for compact quotients of the ball i.e., $\mathbb{B}^n/\Gamma$ where Γ is a discrete subgroup of the automorphism group acting freely. Compact quotients of the ball have HSC < 0.

Example 1.4.17. The holomorphic sectional curvature of a Hermitian metric is *quasi-negative* if HSC ≤ 0 everywhere and there exists $p \in X$ such that

$$\max_{\xi_p \in T_p^{1,0} X \backslash \{0\}} \mathrm{HSC}_g(\xi_p) < 0.$$

An example of a Hermitian metric with quasi-negative (but not negative) HSC was recently constructed by Sarem [Sar23]. We consider the ball quotient $X := \mathbb{B}^n/\Gamma$, where $\mathbb{B}^n$ denotes the unit ball in $\mathbb{C}^n$ and Γ denotes a subgroup of $\mathrm{Aut}(\mathbb{B}^n)$. Let $\overline{X}$ denote the compacitifcation of X. Choosing Γ so that X is non-compact, we may consider the *compactification locus* of X i.e., $\overline{X} \setminus X$. Further imposing that Γ is such that the compactification locus is biholomorphic to complex tori, $\overline{X}$ is referred to as a *toroidal compactification.* The Hermitian metric then constructed on $\overline{X}$ in [Sar23] has quasi-negative HSC on $X \subset \overline{X}$.

We now introduce an important class of Kähler metrics

Definition 1.4.18. Let X be a complex manifold. A Kähler metric g on X is said to be *Kähler–Einstein* if $\mathrm{Ric}_g = \lambda g$ for some constant $\lambda \in \mathbb{R}$.

An example we have already observed is the Fubini–Study metric on $\mathbb{P}^n$, which is Kähler–Einstein with $\lambda = n + 1$. If a compact Kähler manifold admits a Kähler–Einstein metric with $\lambda = 0$, then it is referred to as *Calabi–Yau.* We provide some examples of Calabi–Yau manifolds.

Example 1.4.19. Let $X = \mathbb{C}^n/\Lambda$ denote the n-dimensional complex torus for some lattice $\Lambda \subseteq \mathbb{C}^n$. Endowed with the flat metric, we have that $\mathrm{Ric}_g = 0$, and is therefore Calabi–Yau.

Example 1.4.20. A compact complex surface X ($\dim_{\mathbb{C}} X = 2$) is said to be a *K3 surface* if $h^{0,1} = 0$ and the canonical bundle K_X is holomorphically trivial. It was proven in [Siu83] that every K3 surface admits a Kähler metric, and is therefore Calabi–Yau.

Example 1.4.21. Originally defined by Calabi [Cal79], a *hyperkähler manifold* is a Riemannian manifold (M, g) equipped with three complex structures, I, J, K, which are Kähler with respect to the Riemannian metric g and satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$. The explicit examples of hyperkähler manifolds are rather complicated; the cotangent bundle of projective space $T^*\mathbb{P}^n$ always admits a complete hyperkähler metric [Cal79]. All hyperkähler manifolds are Calabi–Yau (see, e.g., [Zhe00, p. 229]).

To set up the Schwarz lemma, we introduce its primary object of focus.

Definition 2.0.1. Let X be a complex manifold with Dolbeault operators ∂ and $\overline{\partial}$. We refer to the operator $\partial\overline{\partial}$ (or $\sqrt{-1}\partial\overline{\partial}$) as the *complex Hessian.* If X supports a Hermitian metric g, we define the *complex Laplacian* Δ_g as the trace of the complex Hessian with respect to g:

$$\Delta_g f := \operatorname{tr}_g(\sqrt{-1}\partial\overline{\partial} f) = g^{i\overline{j}}\partial_i\partial_{\overline{j}} f$$

where $f \in \mathcal{C}^\infty(X, \mathbb{C})$.

The complex Hessian and Laplacian behave in a familiar manner to the usual Hessian and Laplacian in multivariable calculus.

Proposition 2.0.2 ([SW13, p. 10])**.** Let f be a smooth, real-valued function on a compact Hermitian manifold (X, g) which achieves its maximum at $p \in X$. Then at p, we have

$$df = 0 \qquad \sqrt{-1}\partial\overline{\partial} f \leq 0$$

where for $\alpha = \sqrt{-1}a_{i\overline{j}}dz^i \wedge d\overline{z}^j$, we say that $\alpha \leq 0$ if $a_{i\overline{j}}$ is negative semi-definite.

The above proposition follows from multivariable calculus, wherein the function's first derivative evaluates to 0 and its maximum determines a negative semi-definite Hessian. Further, taking the trace of the complex Hessian with respect to g yields that $\Delta_g f \leq 0$ at the maximum point. We now observe that Δ_g is a second order elliptic operator, meaning that we have access to the maximum principle if the manifold is compact. Hence, if $\Delta_g f \geq 0$ on a compact manifold, then f is constant.

Given a holomorphic map $f : (X, g) \to (Y, h)$ and local holomorphic coordinates $(z^1, \dots, z^n)$ centered at a point $p \in X$, we write $f_i^\alpha := \frac{\partial f^\alpha}{\partial z^i}$, where $f = (f^1, \dots, f^m)$. Further, we use greek indices to denote the holomorphic coordinates on the target manifold. The *energy density* of f is given by

$$|\partial f|^2 := \operatorname{tr}_g(f^*h) = g^{i\overline{j}}h_{\alpha\overline{\beta}}f_i^\alpha\overline{f_j^\beta}.$$

A lemma is of Schwarz-type if it provides a lower bound on the Laplacian of the energy density solely in terms of geometric quantities such as curvature, rank, and dimension [Bro22b].

2.1. **The Kobayashi–Wu Bochner Formula.**

We begin by elaborating on the generalised Bochner formula proved in [Bro22c, Theorem 13.1.1].

Theorem 2.1.1 ([Bro22c, Theorem 13.1.1])**.** Let $(E, h) \to X$ be a Hermitian vector bundle over a complex manifold X. Let ∇ be the Chern connection on E with curvature R. Then for any smooth section $\sigma \in \Gamma(E)$, we have

$$\sqrt{-1}\partial\overline{\partial}|\sigma|_h^2 = 2\mathrm{Re}\langle\nabla^{1,0}\overline{\partial}\sigma, \sigma\rangle - \langle R(\sigma), \sigma\rangle + |\overline{\partial}\sigma|_h^2 + |\nabla^{1,0}\sigma|_h^2,$$

where $\langle\cdot,\cdot\rangle$ is alternative notation for the Hermitian fibre metric h.

Proof. Let $(z^1, \dots, z^n)$ denote local holomorphic coordinates on X and $\{\phi_i\}_{i=1}^r$ be a local holomorphic frame on E. Let $\sigma = \sum_\alpha \sigma^\alpha\phi_\alpha$ be a smooth section of E. We may choose σ^α to be smooth functions on X and $\{\phi_\alpha\}$ to be a local holomorphic frame of E. Denoting the metric components $h_{\alpha\overline{\beta}} := h(\phi_\alpha, \phi_\beta)$

$$\begin{aligned}\nabla_i \nabla_{\overline{j}}(h_{\alpha\overline{\beta}}\sigma^\alpha\overline{\sigma}^\beta) &= h_{\alpha\overline{\beta}}(\nabla_i\nabla_{\overline{j}}\sigma^\alpha)\overline{\sigma}^\beta + h_{\alpha\overline{\beta}}(\nabla_{\overline{j}}\sigma^\alpha)(\nabla_i\overline{\sigma}^\beta)\\ &\quad + h_{\alpha\overline{\beta}}(\nabla_i\sigma^\alpha)(\nabla_{\overline{j}}\overline{\sigma}^\beta) + h_{\alpha\overline{\beta}}\sigma^\alpha(\nabla_i\nabla_{\overline{j}}\overline{\sigma}^\beta).\end{aligned}$$

Observe that the commutation formula

$$\nabla_i\nabla_{\overline{j}}\overline{\sigma}^\beta - \nabla_{\overline{j}}\nabla_i\overline{\sigma}^\beta = -R_{i\overline{j}\overline{\gamma}}{}^{\overline{\beta}}\overline{\sigma}^\gamma$$

then reduces our equation to

$$\begin{aligned}\nabla_i \nabla_{\overline{j}}(h_{\alpha\overline{\beta}}\sigma^\alpha\sigma^\beta) &= h_{\alpha\overline{\beta}}(\nabla_i\nabla_{\overline{j}}\sigma^\alpha)\overline{\sigma}^\beta + h_{\alpha\overline{\beta}}(\nabla_{\overline{j}}\sigma^\alpha)(\nabla_i\overline{\sigma}^\beta)\\ &\quad + h_{\alpha\overline{\beta}}(\nabla_i\sigma^\alpha)(\nabla_{\overline{j}}\overline{\sigma}^\beta) + h_{\alpha\overline{\beta}}\sigma^\alpha(-R_{i\overline{j}\overline{\gamma}}{}^{\beta}\overline{\sigma}^\gamma + \nabla_{\overline{j}}\nabla_i\overline{\sigma}^\beta).\end{aligned}$$

Let $i = j$ and sum over i. If we choose coordinates at a point x_0 such that $h_{\alpha\overline{\beta}}(x_0) = \delta^\alpha_\beta$, then we obtain

$$\begin{aligned}\nabla_i\nabla_{\overline{i}}|\sigma|^2_h &= (\nabla_i\nabla_{\overline{i}}\sigma^\alpha)\overline{\sigma}^\alpha + (\nabla_{\overline{j}}\sigma^\alpha)(\nabla_i\overline{\sigma}^\alpha)\\ &\quad + (\nabla_i\sigma^\alpha)(\nabla_{\overline{i}}\overline{\sigma}^\alpha) + \sigma^\alpha(-R_{i\overline{i}\overline{\gamma}}{}^{\alpha}\overline{\sigma}^\gamma + \nabla_{\overline{i}}\nabla_i\overline{\sigma}^\alpha).\end{aligned}$$

We note that $\overline{(\nabla_i\nabla_{\overline{i}}\sigma^\alpha)\overline{\sigma}^\alpha} = (\nabla_{\overline{i}}\nabla_i\overline{\sigma}^\alpha)\sigma^\alpha$. We then obtain the result

$$\Delta^{(E,h)}|\sigma|^2_h = 2\mathrm{Re}\langle\nabla^{1,0}\overline{\partial}\sigma, \overline{\sigma}\rangle - \langle R(\sigma), \overline{\sigma}\rangle + |\overline{\partial}\sigma|^2_h + |\nabla^{1,0}\sigma|^2_h.$$

□

When considering holomorphic sections of a Hermitian vector bundle, we observe that $\overline{\partial}\sigma = 0$. This yields the pivotal Bochner formula proven by Kobayashi–Wu [KW70].

Theorem 2.1.2 (Kobayashi–Wu Bochner Formula, [KW70])**.** Let $E \to X$ be a Hermitian vector bundle over a compact complex manifold X with Hermitian fibre metric h. Let ∇ be the Chern connection on E. Then, for any holomorphic section $\sigma \in H^0(E)$, we have

$$\sqrt{-1}\partial\overline{\partial}|\sigma|^2_h = |\nabla^{1,0}\sigma|^2_h - \langle R^E(\sigma), \overline{\sigma}\rangle$$

where R^E is the curvature of the connection.

In particular, we may apply the Kobayashi–Wu Bochner formula to holomorphic maps. Let $f : (X, g) \to (Y, h)$ be a holomorphic map between Hermitian manifolds. If we consider $E = (T^{1,0}X)^* \otimes f^*T^{1,0}Y$, then ∂f is a holomorphic section of E. We may also define a bundle metric on E given by $H := g^{-1} \otimes f^*h$. Note that curvature splits additively across the tensor bundle, and the dualising of $T^{1,0}X$ contributes a negative sign. Further, it commutes with pullbacks (see, e.g., [GH94]):

$$R^{(T^{1,0}X)^*\otimes f^*T^{1,0}Y} = -R^{T^{1,0}X} \otimes \mathrm{Id} + \mathrm{Id} \otimes f^*R^{T^{1,0}Y}$$

Hence, we obtain

$$\partial\overline{\partial}|\partial f|^2_H = \langle\nabla\partial f, \overline{\nabla\partial f}\rangle_H + \langle(R^{T^{1,0}X} \otimes \mathrm{Id})(\partial f), \overline{\partial f}\rangle_H - \langle(\mathrm{Id} \otimes f^*R^{T^{1,0}Y}), \overline{\partial f}\rangle_H. \tag{1}$$

Tracing over Eq. (1) with respect to the source metric, we obtain the Chern–Lu identity. If it is traced over with respect to the target metric, we call the identity Aubin–Yau.

2.2. **The Chern–Lu Schwarz Lemma.** Tracing over Eq. (1) with respect to our source metric, we obtain the following theorem proven in [Lu68].

Theorem 2.2.1 (Chern–Lu Identity, [Lu68])**.** Let $f : (X, g) \to (Y, h)$ be a holomorphic map between Hermitian manifolds. Then in local holomorphic coordinates $\{z_i\}$ and $\{w_\alpha\}$ on X and Y respectively, we have the identity

$$\Delta_g |\partial f|^2 = \left(g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}} \right) g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f^\alpha_p \overline{f^\beta_q} + |\nabla \partial f|^2 - R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} \left(g^{i\bar{j}} f^\alpha_i \overline{f^\beta_j} \right) \left(g^{p\bar{q}} f^\gamma_p \overline{f^\delta_q} \right).$$

The first term in this sum is controlled by assuming a lower bound on the second Chern Ricci curvature. Indeed, suppose that $\mathrm{Ric}^{(2)}_g \geq -Cg$ for some $C > 0$. Since $g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}} = \mathrm{Ric}^{(2)}_{k\bar{\ell}}$, we obtain

$$g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f^\alpha_p \overline{f^\beta_q} \geq -C g_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f^\alpha_p \overline{f^\beta_q} = -C g^{p\bar{q}} h_{\alpha\bar{\beta}} f^\alpha_p \overline{f^\beta_q} = -C |\partial f|^2.$$

The second term is the *second fundamental form* of f, which is always non-negative. We consider the holomorphic vector bundle $E := \Lambda^{1,0} X \otimes f^* T^{1,0} Y$, which we refer to as the *twisted cotangent bundle*. Given Chern connections $\widehat{\nabla}, \widetilde{\nabla}$ on $T^{1,0}X$ and $T^{1,0}Y$ respectively, we define the connection ∇ as the tensor product connection of $\widehat{\nabla}^*$ (the dual connection of $\widehat{\nabla}$) and $f^*\widetilde{\nabla}$ (the pullback connection of $\widetilde{\nabla}$) on E. Computing this expression in a local coordinate frame, we have that

$$(\nabla_k \partial f)^\alpha_\ell = f^\alpha_{k\ell} + \Gamma^\alpha_{\gamma\rho} f^\gamma_k f^\rho_\ell - \Gamma^m_{k\ell} f^\alpha_m.$$

The last term however is rather difficult to observe. Choose coordinates on M and N such that $g_{i\bar{j}} = \delta^i_j, h_{\alpha\bar{\beta}} = \delta^\alpha_\beta$, and $f^\alpha_i = \lambda_i \delta^\alpha_i$, where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r \geq \lambda_{r+1} \geq 0$, where r is the rank of $\partial f = (f^\alpha_i)$. This is the principle value decomposition of a matrix. Then, we obtain

$$g^{k\bar{\ell}} g^{i\bar{j}} R^h_{\gamma\bar{\delta}\alpha\bar{\beta}} f^\alpha_i \overline{f^\beta_j} f^\gamma_k \overline{f^\delta_\ell} = \delta^\ell_k \delta^j_i R^h_{\gamma\bar{\delta}\alpha\bar{\beta}} \lambda_i \delta^\alpha_i \lambda_j \delta^\beta_j \lambda_k \delta^\gamma_k \lambda_\ell \delta^\delta_\ell = \sum_{\alpha,\gamma} R^h_{\gamma\bar{\gamma}\alpha\bar{\alpha}} \lambda^2_\gamma \lambda^2_\alpha.$$

We narrow our focus to the case where $f : (C, g) \to (X, h)$ is a holomorphic curve (or more generally, if f has rank one). In local coordinates, $\partial f = (f^\alpha_1) := \left(\frac{\partial f^1}{\partial z}, \ldots, \frac{\partial f^n}{\partial z} \right)$ and is a row vector at each point. Computing the last term with respect to this map yields

$$-R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} \left(g^{i\bar{j}} f^\alpha_i \overline{f^\beta_j} \right) \left(g^{p\bar{q}} f^\gamma_p \overline{f^\delta_q} \right) = -R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} \left(g^{1\bar{1}} f^\alpha_1 \overline{f^\beta_1} \right) \left(g^{1\bar{1}} f^\gamma_1 \overline{f^\delta_1} \right) = -g^{-1} \otimes \mathrm{HSC}_h(\partial f),$$

and is therefore controlled by the holomorphic sectional curvature. This immediately yields the following corollary.

Corollary 2.2.2. Let (X, g) be a Hermitian manifold. If $\mathrm{HSC}_g \leq 0$, then X has no rational curves.

Proof. Let $f : (\mathbb{P}^1, g) \to (X, h)$ be a holomorphic map, where g is the Fubini–Study metric. The Fubini–Study metric is Kähler with $\mathrm{Ric}_g = 2g$. Since $\mathrm{HSC}_h \leq 0$, the Chern–Lu calculation yields

$$\Delta_g |\partial f|^2 \geq 2 g_{1\bar{1}} g^{1\bar{1}} g^{1\bar{1}} f^\gamma_1 \overline{f^\delta_1} h_{\gamma\bar{\delta}} = 2 g^{1\bar{1}} g^{1\bar{1}} f^\gamma_1 \overline{f^\delta_1} h_{\gamma\bar{\delta}} = 2|\partial f|^2.$$

Since $\mathbb{P}^1$ is compact, we may integrate over both sides and apply the divergence theorem to yield that any holomorphic map $f : \mathbb{P}^1 \to X$ is constant i.e., there are no rational curves on X. $\square$

Remark 2.2.3. The above corollary does not require any assumptions on the completeness of X.

This in particular implies that there are no rational curves on a torus. For holomorphic maps of higher rank, further work is required to establish suitable bounds. In the Kähler setting, Royden [Roy80] demonstrated that a sign on the holomorphic sectional curvature of h forces control of the last term. This amounts to a linear algebraic result, referred to as *Royden's trick* in the literature (see e.g., [Bro22a]).

Theorem 2.2.4 (Royden's Trick, [Roy80])**.** Let $\xi_1, \dots, \xi_n$ be orthogonal tangent vectors on a Hermitian manifold (X, g). Let $\mathcal{S}(\xi, \overline{\eta}, \zeta, \overline{\omega})$ be a symmetric bi-Hermitian form, i.e.,

$$\mathcal{S}(\xi, \overline{\eta}, \zeta, \overline{\omega}) = \mathcal{S}(\zeta, \overline{\eta}, \xi, \overline{\omega}), \qquad \mathcal{S}(\eta, \overline{\xi}, \omega, \overline{\zeta}) = \overline{\mathcal{S}}(\xi, \overline{\eta}, \zeta, \overline{\omega}).$$

If $\mathcal{S}(\xi, \overline{\xi}, \xi, \overline{\xi}) \leq \kappa \|\xi\|^4$, then

$$\sum_{\alpha,\beta} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\alpha, \overline{\xi}_\alpha) \leq \kappa \frac{n+1}{2n} \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2$$

Proof. Consider $A \in \mathbb{Z}_4^n$, denoted $A = (\varepsilon^1, \dots, \varepsilon^n)$ where $\varepsilon_\alpha^4 = 1$ for each α. We have that $\varepsilon_\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$. Set $\xi_A := \sum_\alpha \varepsilon_\alpha \xi_\alpha$. Then $\|\xi_A\|^2 = \sum_\alpha \|\xi_\alpha\|^2$, and so

$$\mathcal{S}(\xi_A, \overline{\xi}_A, \xi_A, \overline{\xi}_A) \leq \kappa \|\xi_A\|^2 = \kappa \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2.$$

Summing both sides over $A \in \mathbb{Z}_4^n$, we obtain

$$\begin{aligned}
\kappa \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2 &\geq \frac{1}{4^n} \sum_{A \in \mathbb{Z}_4^n} \mathcal{S}(\xi_A, \overline{\xi}_A, \xi_A, \overline{\xi}_A) \\
&= \frac{1}{4^n} \sum_{\alpha,\beta,\gamma,\delta=1}^{n} \varepsilon_\alpha \overline{\varepsilon}_\beta \varepsilon_\gamma \overline{\varepsilon}_\delta \mathcal{S}(\xi_\alpha, \overline{\xi}_\beta, \xi_\gamma, \overline{\xi}_\delta) \\
&= \sum_\alpha \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\alpha, \overline{\xi}_\alpha) + \sum_{\alpha \neq \gamma} (\mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) + \mathcal{S}(\xi_\alpha, \overline{\xi}_\gamma, \xi_\gamma, \overline{\xi}_\alpha)).
\end{aligned}$$

By the symmetry of $\mathcal{S}$, we have

$$\sum_\alpha \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\alpha, \overline{\xi}_\alpha) + 2 \sum_{\alpha \neq \gamma} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) \leq \kappa \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2.$$

Adding $\sum_\alpha \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\alpha, \overline{\xi}_\alpha)$ to both sides and applying the assumption that $\mathcal{S}(\xi, \overline{\xi}, \xi, \overline{\xi}) \leq \kappa \|\xi\|^4$, we then obtain

$$2 \sum_{\alpha,\gamma} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) \leq \kappa \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2 + \kappa \sum_\alpha \|\xi_\alpha\|^4.$$

Now suppose that $\kappa \leq 0$. Since $(\sum_\alpha \|\xi_\alpha\|^2)^2 \leq n \sum_\alpha \|\xi_\alpha\|^4$ by the Cauchy-Schwarz inequality, we see that

$$\kappa \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2 \geq \kappa n \sum_\alpha \|\xi_\alpha\|^4.$$

Hence,

$$2 \sum_{\alpha,\gamma} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) \leq \kappa \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2 + \frac{\kappa}{n} \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2$$

$$\sum_{\alpha,\gamma} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) \leq \kappa \frac{n+1}{2n} \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2$$

as desired. □

Applying this trick to our mysterious term (in the simpler coordinates), we obtain a bound in terms of the holomorphic sectional curvature. For a general Hermitian metric, however, Royden's argument cannot be used. To summarise, we see that if g is a Hermitian metric with $\mathrm{Ric}_g^{(2)} \geq Cg$ and h is a Kähler metric with $\mathrm{HSC}_h \leq -\kappa$, then

$$\Delta_g |\partial f|^2 = |\nabla \partial f|^2 + C|\partial f|^2 + \kappa \frac{r+1}{r} |\partial f|^4 \geq C|\partial f|^2 + \kappa \frac{r+1}{r} |\partial f|^4.$$

Hence, we obtain Royden's Schwarz lemma.

Theorem 2.2.5 (Royden Schwarz lemma, [Roy80])**.** Let $f : (X, g) \to (Y, h)$ be a holomorphic map from a Hermitian manifold to a Kähler manifold. Suppose $\mathrm{Ric}_g^{(2)} \geq Cg$ and $\mathrm{HSC}_h \leq -\Lambda_0 < 0$ for constants C, Λ_0. Then

$$\Delta_g |\partial f|^2 \geq C|\partial f|^2 + \frac{r+1}{r} \Lambda_0 |\partial f|^4$$

where $r = \mathrm{rank}(\partial f)$.

So far, suitable curvature controls in the non-Kähler category are out of reach. This prompted the consideration of the *real bisectional curvature* in [YZ19].

Definition 2.2.6. The *real bisectional curvature* of a Hermitian metric g is defined by

$$\mathrm{RBC}_g(\xi) := \frac{1}{|\xi|_g^2} \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} \xi^{i\bar{j}} \xi^{k\bar{\ell}},$$

where ξ is a non-negative Hermitian (1,1)-tensor.

A sign on the real bisectional curvature, $\mathrm{RBC}_g < 0$ for instance, forces that same sign on the holomorphic sectional curvature [YZ19]. If our metric is Kähler, then the converse also holds. When considering this curvature term, we obtain a refined version of the Chern–Lu inequality for Hermitian metrics.

Theorem 2.2.7 (Yang–Zheng Schwarz lemma, [YZ19, Theorem 4.3])**.** Let $f : (X, g) \to (Y, h)$ be a holomorphic map between Hermitian manifolds. Assume $\mathrm{Ric}_g^{(2)} \geq C_1 g + C_2 f^* h$ for constants C_1, C_2. Assume that $\mathrm{RBC}_h \leq -\kappa \leq 0$ for some constant κ. Then,

$$\Delta_g |\partial f|^2 \geq C_1 |\partial f|^2 + \frac{(C_2 + \kappa)}{r} |\partial f|^4 + |\nabla \partial f|^2$$

where r denotes the rank of ∂f.

Proof. By Theorem 2.2.1, we have the following identity:

$$\Delta_g |\partial f|^2 = \left(g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}} \right) g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} + |\nabla \partial f|^2 - R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} \left(g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} \right) \left(g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta} \right).$$

Observe $g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}} = \mathrm{Ric}^{(2)}_{k\bar{\ell}}$, the second Chern Ricci curvature of g. We briefly adopt the notation $\mathrm{Ric}^g_{k\bar{\ell}}$ to denote the $k\bar{\ell}$-component of $\mathrm{Ric}_g^{(2)}$. From the assumptions on the curvature, we have

$$\mathrm{Ric}^g_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} \quad \geq \quad C_1 g^{p\bar{q}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} + C_2 g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} h_{\gamma\bar{\delta}} f_p^\alpha \overline{f_q^\beta} f_k^\gamma \overline{f_\ell^\delta}. \tag{2}$$

The first term on the right-hand side of (2) is clearly $C_1 \mathrm{tr}_g(f^*h)$. For the second term, choose coordinates such that $g_{i\bar{j}} = \delta^i_j, h_{\alpha\bar{\beta}} = \delta^\alpha_\beta$ and $f^\alpha_i = \lambda_i \delta^\alpha_i$, where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r \geq \lambda_{r+1} \geq 0$, where r is the rank of $\partial f = (f^\alpha_i)$. This is the principle value decomposition of a matrix. In these coordinates, we see that

$$g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} h_{\gamma\bar{\delta}} f^\alpha_p \overline{f^\beta_q} f^\gamma_k \overline{f^\delta_\ell} \quad = \quad \delta^q_k \delta^\ell_p \delta^\beta_\alpha \delta^\delta_\gamma \lambda_p \delta^\alpha_p \lambda_q \delta^\beta_q \lambda_k \delta^\gamma_k \lambda_\ell \delta^\delta_\ell \; = \sum_\alpha \lambda^4_\alpha.$$

Then $\left(\sum_\alpha \lambda^2_\alpha\right)^2 \leq r \sum_\alpha \lambda^4_\alpha$ by the Cauchy–Schwarz inequality. Indeed,

$$\left(\sum_{i=1}^r \lambda_i^2 \cdot 1\right)^2 \leq r \left(\sum_{i=1}^r \lambda_i^4\right)$$

Hence, we obtain

$$\mathrm{Ric}^g_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f^\alpha_p \overline{f^\beta_q} \quad \geq \quad C_1 |\partial f|^2 + \frac{C_2}{r} |\partial f|^4.$$

These coordinates will also be useful for analyzing the remaining curvature term. Indeed, in these coordinates, we see that

$$g^{k\bar{\ell}} g^{i\bar{j}} R^h_{\gamma\bar{\delta}\alpha\bar{\beta}} f^\alpha_i \overline{f^\beta_j} f^\gamma_k \overline{f^\delta_\ell} \quad = \quad \delta^\ell_k \delta^j_i R^h_{\gamma\bar{\delta}\alpha\bar{\beta}} \lambda_i \delta^\alpha_i \lambda_j \delta^\beta_j \lambda_k \delta^\gamma_k \lambda_\ell \delta^\delta_\ell \; = \; R^h_{\gamma\bar{\gamma}\alpha\bar{\alpha}} \lambda^2_\gamma \lambda^2_\alpha.$$

Since $\mathrm{RBC}_h \leq -\kappa \leq 0$, we have

$$R^h_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \lambda^2_\alpha \lambda^2_\gamma \quad \leq \quad -\kappa \sum_\alpha \lambda^4_\alpha \; \leq \; -\frac{\kappa}{r} \left(\sum_\alpha \lambda^2_\alpha\right)^2 \quad = \quad -\frac{\kappa}{r} |\partial f|^4.$$

We therefore have that $g^{k\bar{\ell}} g^{i\bar{j}} R^h_{\gamma\bar{\delta}\alpha\bar{\beta}} f^\alpha_i \overline{f^\beta_j} f^\gamma_k \overline{f^\delta_\ell} \leq -\frac{\kappa}{r} |\partial f|^4$. Combining the above inequalities, we obtain

$$\Delta_g |\partial f|^2 \geq C_1 |\partial f|^2 + \frac{(C_2 + \kappa)}{r} |\partial f|^4 + |\nabla \partial f|^2.$$

□

One particular avenue exploited by Broder–Pulemotov [BP23] is the Schwarz lemma applied to Hermitian metrics of vanishing second Chern Ricci curvature. Using the Yang–Zheng schwarz lemma, they obtain the following theorem.

Theorem 2.2.8 ([BP23, Theorem 1.1])**.** Let (X, g) be a compact Hermitian manifold with $\mathrm{RBC}_g \leq 0$. If the equality $\mathrm{Ric}^{(2)}_h = 0$ holds for some Hermitian metric h on X, then h has the same Chern connection as g. If there is a point where $\mathrm{RBC}_h < 0$, then there are no metrics on X with vanishing second Chern Ricci curvature.

Proof. Let $f : (X, g) \to (X, h)$ be a holomorphic map. In coordinates, we may write $|\nabla \partial f|^2$ as

$$(\nabla_k \partial f)^\alpha_\ell = f^\alpha_{k\ell} + \Gamma^\alpha_{\gamma\rho}(h) f^\gamma_k f^\rho_\ell - \Gamma^m_{k\ell}(g) f^\alpha_m,$$

where $\Gamma^m_{k\ell}(g), \Gamma^\alpha_{\gamma\rho}(h)$ denote the Christoffel symbols (of the Chern connection) of g and h respectively. Since $\mathrm{Ric}^{(2)}_h = 0$ and $\mathrm{RBC}_h \leq 0$, the Chern–Lu inequality yields

$$\Delta_h |\partial f|^2 \geq |\nabla \partial f|^2.$$

Since X is compact, the maximum principle implies that $|\partial f|^2$ is constant, and therefore $\nabla \partial f = 0$. Let $f = \mathrm{id}$. Hence,

$$0 = (\nabla_k \partial f)^\alpha_\ell = f^\alpha_{k\ell} + \Gamma^\alpha_{\gamma\rho}(h)\delta^\gamma_k \delta^\rho_\ell - \Gamma^m_{k\ell}(g)\delta^\alpha_m = \Gamma^m_{k\ell}(h) - \Gamma^m_{k\ell}(g),$$

which implies g and h have the same Chern connection. If $\mathrm{RBC}_h < 0$ at same point, we arrive at the same contradiction, meaning that there are no second Chern Ricci flat metrics on compact manifolds with $\mathrm{RBC}_h < 0$ at a point. □

Corollary 2.2.9 ([BP23, Corollary 1.2])**.** Let (X, h) be a compact Kähler manifold with $\mathrm{HSC}_h \leq 0$. Then every Hermitian metric g on X satisfying $\mathrm{Ric}^{(2)}_g = 0$ is Kähler. If such a metric exists, the manifold X is Calabi–Yau.

Proof. Let $f : (X, g) \to (X, h)$ be a holomorphic map. Royden's trick then implies that $\mathrm{RBC}_h \leq 0$. Hence, we may apply Theorem 2.2.8 to obtain the equality of Chern connections in the $\mathrm{HSC}_h \leq 0$ case, and non-existence if $\mathrm{HSC}_h < 0$. Therefore, since the Chern connection of h has vanishing torsion (as it is Kähler), the Chern connection of g must also have vanishing torsion. This implies that g is Kähler. Further, as $\mathrm{Ric}_g = 0$, X is Calabi–Yau. □

If our metric is Kähler, there is only one Ricci curvature. Hence, the class of Hermitian metrics with vanishing second Chern Ricci curvature manifests itself as Calabi–Yau in the Kähler case.

Corollary 2.2.10. Calabi–Yau manifolds admit no Kähler metrics with $\mathrm{HSC} < 0$.

Proof. Let (X, g) denote a Calabi–Yau manifold. Assume there exists a Kähler metric h on X with $\mathrm{HSC}_h < 0$. Then, we have that $\mathrm{RBC}_h \leq -\kappa < 0$ where $\kappa > 0$. Let $\mathrm{Id} : (X, g) \to (X, h)$ denote the identity map. The Chern–Lu inequality then gives us

$$\Delta_g |\partial \mathrm{Id}|^2 \geq \frac{\kappa}{r} |\partial \mathrm{Id}|^4 > 0.$$

Integrating both sides then yields a contradiction. □

A further development to Chern–Lu Schwarz lemma was considered by Broder–Stanfield [BS23], where they turned their attention to the Hessian $|\nabla \partial f|^2$. As with matrices, the Hessian may be broken into its symmetric and skew-symmetric parts, denoted $\mathrm{Sym}(\nabla \partial f)$ and $\mathrm{Skew}(\nabla \partial f)$ respectively. This resulted in the discovery of the following curvature.

Definition 2.2.11 (Broder–Stanfield, [BS23])**.** The *tempered real bisectional curvature* of a Hermitian metric g is defined by

$$\mathrm{RBC}^\tau_g(\xi) := \frac{1}{|\xi|^2_g} \sum_{i,j,k,\ell} \left(R_{i\bar{j}k\bar{\ell}} \xi^{i\bar{j}} \xi^{k\bar{\ell}} - \frac{1}{4} \sum_{p,q} T^p_{ik} \overline{T^q_{j\ell}} \xi^{i\bar{j}} \xi^{k\bar{\ell}} g_{p\bar{q}} \right)$$

where ξ is a non-negative Hermitian $(1,1)$-tensor, and T is the torsion of the Chern connection.

This novel curvature term provides control over the holomorphic sectional curvature of a non-trivial class of Hermitian (non-Kähler) metrics. In particular, if the metric is pluriclosed, negative holomorphic sectional curvature implies that the tempered real bisectional curvature is negative.

Proposition 2.2.12. Let (X, g) be a pluriclosed manifold. Then HSC_g and RBC^τ_g are comparable in the sense that they always have the same sign.

Proof. We note that the pluriclosed condition is equivalent to

$$R_{i\bar{j}k\bar{\ell}} - R_{k\bar{j}i\bar{\ell}} - R_{i\bar{\ell}k\bar{j}} + R_{k\bar{\ell}i\bar{j}} = \sum_{p,q} T_{ik}^{p} \overline{T_{j\ell}^{q}} h_{p\bar{q}}. \tag{3}$$

This is a consequence of the first Bianchi identity for curvature forms (see e.g., [RZ22, p. 3]). Observe that from Eq. (3), we have

$$\begin{aligned}
\sum_{i,j,k,\ell,p,q} T_{ik}^{p} \overline{T_{j\ell}^{q}} \xi^{i\bar{j}} \xi^{k\bar{\ell}} h_{p\bar{q}} &= \sum_{i,j,k,\ell} (R_{i\bar{j}k\bar{\ell}} - R_{k\bar{j}i\bar{\ell}} - R_{i\bar{\ell}k\bar{j}} + R_{k\bar{\ell}i\bar{j}}) \xi^{i\bar{j}} \xi^{k\bar{\ell}} \\
&= \sum_{i,j,k,\ell} (R_{i\bar{j}k\bar{\ell}} - R_{i\bar{\ell}k\bar{j}}) \xi^{i\bar{j}} \xi^{k\bar{\ell}} + \sum_{i,j,k,\ell} (R_{k\bar{\ell}i\bar{j}} - R_{k\bar{j}i\bar{\ell}}) \xi^{i\bar{j}} \xi^{k\bar{\ell}} \\
&= \sum_{i,j,k,\ell} (R_{i\bar{j}k\bar{\ell}} - R_{i\bar{\ell}k\bar{j}}) \xi^{i\bar{j}} \xi^{k\bar{\ell}} + \sum_{i,j,k,\ell} (R_{i\bar{j}k\bar{\ell}} - R_{i\bar{\ell}k\bar{j}}) \xi^{k\bar{\ell}} \xi^{i\bar{j}} \\
&= 2 \sum_{i,j,k,\ell} (R_{i\bar{j}k\bar{\ell}} - R_{i\bar{\ell}k\bar{j}}) \xi^{i\bar{j}} \xi^{k\bar{\ell}}
\end{aligned}$$

We compute $\mathrm{RBC}_g^{\tau}(\xi)$ for when g is pluriclosed.

$$\begin{aligned}
\mathrm{RBC}_g^{\tau}(\xi) &= \frac{1}{|\xi|_g^2} \sum_{i,j,k,\ell} \left(R_{i\bar{j}k\bar{\ell}} \xi^{i\bar{j}} \xi^{k\bar{\ell}} - \frac{1}{4} \sum_{p,q} T_{ik}^{p} \overline{T_{j\ell}^{q}} \xi^{i\bar{j}} \xi^{k\bar{\ell}} g_{p\bar{q}} \right) \\
&= \frac{1}{|\xi|_g^2} \sum_{i,j,k,\ell} \left(R_{i\bar{j}k\bar{\ell}} - \frac{1}{2} R_{i\bar{j}k\bar{\ell}} + \frac{1}{2} R_{i\bar{\ell}k\bar{j}} \right) \xi^{i\bar{j}} \xi^{k\bar{\ell}} \\
&= \frac{1}{2|\xi|_g^2} \sum_{i,j,k,\ell} \left(R_{i\bar{j}k\bar{\ell}} + R_{i\bar{\ell}k\bar{j}} \right) \xi^{i\bar{j}} \xi^{k\bar{\ell}}.
\end{aligned}$$

This curvature term may be considered by the *altered holomorphic sectional curvature*, defined in [BT24] as

$$\widetilde{\mathrm{HSC}}_g(\xi) := \frac{1}{|\xi|_g^2} \sum_{i,j,k,\ell} (R_{i\bar{j}k\bar{\ell}} + R_{i\bar{\ell}k\bar{j}}) \xi^{i\bar{j}} \xi^{k\bar{\ell}},$$

where ξ is a non-negative Hermitian (1,1)-tensor. Hence, $\mathrm{RBC}_g^{\tau} = \frac{1}{2}\widetilde{\mathrm{HSC}}_g$. This curvature term is also considered implicitly in [YZ19], where it is shown to have the same sign as the holomorphic sectional curvature (see, e.g., [YZ19, p. 5]). This demonstrates that RBC_g^{τ} and HSC_g have the same sign. □

This yields the following extension of the Royden Schwarz lemma.

Theorem 2.2.13 (Broder–Stanfield Schwarz lemma, [BS23, Corollary 3.1])**.** Let $f : (X, g) \to (Y, h)$ be a holomorphic map of rank r from a compact Kähler manifold to a pluriclosed manifold. Suppose there are constants $C_1, C_2 \in \mathbb{R}$ and $\Lambda_0 \in \mathbb{R}_{\geq 0}$ such that $\mathrm{Ric}_g \geq C_1 g + C_2 f^* h$, and $\mathrm{HSC}_g \leq -\Lambda_0 \leq 0$. Then,

$$\Delta_g |\partial f|^2 \geq C_1 |\partial f|^2 + \left(\frac{\Lambda_0 + C_2}{r} \right) |\partial f|^4$$

Proof. We begin with the identity afforded to us by Theorem 2.2.1:

$$\Delta_g |\partial f|^2 = \mathrm{Ric}^g_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^{\alpha} \overline{f_q^{\beta}} + |\nabla \partial f|^2 - R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} \left(g^{i\bar{j}} f_i^{\alpha} \overline{f_j^{\beta}} \right) \left(g^{p\bar{q}} f_p^{\gamma} \overline{f_q^{\delta}} \right).$$

We consider breaking apart the hessian $|\nabla\partial f|^2$ into its symmetric and skew-symmetric parts. The skew-symmetric part of $|\nabla\partial f|^2$, written as $\mathrm{Skew}(\nabla\partial f)$, is given in a local frame by

$$\mathrm{Skew}(\nabla\partial f)_{ij}^{\alpha} = \frac{1}{2}((\nabla_i \partial f)_j^{\alpha} - (\nabla_j \partial f)_i^{\alpha}),$$

which follows from the decomposition of a matrix into its symmetric and skew symmetric parts. Computing this explicitly, we obtain

$$\begin{aligned}
\mathrm{Skew}(\nabla\partial f)_{ij}^{\alpha} &= \frac{1}{2}((\nabla_i \partial f)_j^{\alpha} - (\nabla_j \partial f)_i^{\alpha}) \\
&= \frac{1}{2}(f_i^{\gamma}\Gamma_{\gamma\rho}^{\alpha} f_j^{\rho} - \Gamma_{ij}^{p} f_p^{\alpha} - (f_j^{\gamma}\Gamma_{\gamma\rho}^{\alpha} f_i^{\rho} - \Gamma_{ji}^{p} f_p^{\alpha})) \\
&= \frac{1}{2}(f_i^{\gamma} f_k^{\rho} T_{\gamma\rho}^{\alpha} + T_{ji}^{\rho} f_p^{\alpha}) \\
&= \frac{1}{2}(f_i^{\gamma} f_k^{\rho} T_{\gamma\rho}^{\alpha} - T_{ij}^{\rho} f_p^{\alpha}).
\end{aligned}$$

Since g is Kähler, we have that its torsion vanishes. Hence, $\mathrm{Skew}(\nabla\partial f)_{ij}^{\alpha} = \frac{1}{2}(f_i^{\gamma} f_k^{\rho} T_{\gamma\rho}^{\alpha})$. Further, the skew-symmetric and symmetric parts of $|\nabla\partial f|^2$ are orthogonal, meaning that

$$|\nabla\partial f|^2 = \frac{1}{4}\sum_{i,j,\alpha} |f_i^{\gamma} f_j^{\rho} T_{\gamma\rho}^{\alpha}|^2 + |\mathrm{Sym}(\nabla\partial f)|^2,$$

where $\mathrm{Sym}(\nabla\partial f)$ denotes the symmetric part of $\nabla\partial f$. It is a straightforward tensor calculation to see that

$$|f_i^{\gamma} f_j^{\rho} T_{\gamma\rho}^{\alpha}|^2 = T_{\alpha\gamma}^{\rho}\overline{T_{\beta\delta}^{\sigma}} h_{\rho\overline{\sigma}}(g^{i\overline{j}} f_i^{\alpha}\overline{f_j^{\beta}})(g^{p\overline{q}} f_p^{\gamma}\overline{f_q^{\delta}}).$$

Combining the above results, we obtain

$$\Delta_g|\partial f|^2 = |\mathrm{Sym}(\nabla\partial f)|^2 + \mathrm{Ric}_{k\overline{\ell}}^{g} g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f_p^{\alpha}\overline{f_q^{\beta}} - \left(R_{\alpha\overline{\beta}\gamma\overline{\delta}}^{h} - \frac{1}{4}T_{\alpha\gamma}^{\rho}\overline{T_{\beta\delta}^{\sigma} h_{\rho\overline{\sigma}}}\right)(g^{i\overline{j}} f_i^{\alpha}\overline{f_j^{\beta}})(g^{p\overline{q}} f_p^{\gamma}\overline{f_q^{\delta}}).$$

Since $|\mathrm{Sym}(\nabla\partial f)|^2 \geq 0$, we may disregard it. Using similar reasoning as in Theorem 2.2.7, the bounds on the Ricci curvature yield

$$\mathrm{Ric}_{k\overline{\ell}}^{g} g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f_p^{\alpha}\overline{f_q^{\beta}} \geq C_1|\partial f|^2 + \frac{C_2}{r}|\partial f|^4.$$

Since h is pluriclosed and $\mathrm{HSC}_h \leq 0$, then $\mathrm{RBC}_h^{\tau} \leq -\Lambda_0 \leq 0$. These bounds on RBC_h^{τ} then yield

$$\left(R_{\alpha\overline{\beta}\gamma\overline{\delta}}^{h} - \frac{1}{4}T_{\alpha\gamma}^{\rho}\overline{T_{\beta\delta}^{\sigma} h_{\rho\overline{\sigma}}}\right)(g^{i\overline{j}} f_i^{\alpha}\overline{f_j^{\beta}})(g^{p\overline{q}} f_p^{\gamma}\overline{f_q^{\delta}}) \leq \frac{-\Lambda_0}{r}|\partial f|^4.$$

Combining all above inequalities then yields our result. □

An immediate corollary of the Broder–Stanfield Schwarz lemma is the following.

Corollary 2.2.14. Every holomorphic map from a compact Kähler manifold with $\mathrm{Ric}_g \geq 0$ to a pluriclosed manifold with $\mathrm{HSC}_h \leq -\Lambda_0 < 0$ is constant.

2.3. **The Aubin–Yau Schwarz Lemma.** So far, all Schwarz lemmas have come from tracing over the Hessian $\partial\overline{\partial}|\partial f|^2$ with respect to the source metric. If we further assume that f is biholomorphic onto its image, we may trace over this Hessian with respect to the target metric. Such estimates were first considered in [Aub78], [Yau78], and are therefore referred to as Aubin–Yau Schwarz lemmas.

Theorem 2.3.1 (Aubin–Yau Identity, [Aub78],[Yau78])**.** Let $f : (M, g) \to (N, h)$ be a holomorphic map between Hermitian manifolds which is biholomorphic onto its image. Then in local holomorphic coordinates $\{z_i\}$ and $\{w_\alpha\}$ on M and N respectively, we have the identity

$$\Delta_h |\partial f|^2 = |\nabla \partial f|^2 - \left(h^{\gamma\overline{\delta}} R^h_{\gamma\overline{\delta}\alpha\overline{\beta}} \right) g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta} + h^{\gamma\overline{\delta}} g^{i\overline{q}} g^{p\overline{j}} R^g_{k\overline{\ell}p\overline{q}} h_{\alpha\overline{\beta}} f_i^\alpha \overline{f_j^\beta} (f^{-1})^k_\gamma \overline{(f^{-1})^\ell_\delta}.$$

The identity, first considered in [Aub78], [Yau78], may be analysed in a similar fashion to the Chern–Lu identity. The first term is easily controllable, while the second term is controlled by $\text{Ric}^{(2)}_h$. Again, we are faced with a foreign curvature term in the third summand. In the Kähler case, Yau observed that it may be controlled by the holomorphic bisectional curvature.

Theorem 2.3.2 (Aubin–Yau Identity, [Aub78],[Yau78])**.** Let $f : (M, g) \to (N, h)$ be a holomorphic map, which is biholomorphic onto its image. Assume that $\text{HBC}_g \geq -\kappa$ and $\text{Ric}^{(2)}_h \leq C_1 h + C_2 (f^{-1})^* g$ for constants κ, C_1, C_2. Then,

$$\Delta_h |\partial f|^2 \geq |\nabla \partial f|^2 + C_1 |\partial f|^2 - nC_2 - \kappa$$

where $n = \dim_{\mathbb{C}}(M)$.

The curvature conditions for Hermitian metrics are not well understood prompting, Broder [Bro22b] to consider a novel curvature term:

Definition 2.3.3. The *Schwarz bisectional curvature* of a Hermitian metric g is defined by

$$\text{SBC}_g(\xi) := \sum_{i,j,k,\ell} R_{i\overline{j}k\overline{\ell}} \xi^{i\overline{j}} (\xi^{-1})^{k\overline{\ell}},$$

where ξ is a positive-definite Hermitian (1,1)-tensor.

The Schwarz bisectional curvature provides a Hermitian analogue for the Aubin–Yau Schwarz lemma. Although little is known about the Schwarz bisectional curvature, it is dominated by the holomorphic bisectional curvature. Hence, all Hermitian symmetric spaces have non-negative SBC. Adopting this curvature term then yields a Hermitian Aubin–Yau inequality.

Theorem 2.3.4 (Broder Schwarz lemma, [Bro22b, Theorem 1.3])**.** Let $f : (X, g) \to (Y, h)$ be a holomorphic map between Hermitian manifolds, which is biholomorphic onto its image. Assume $\text{SBC}_g \geq -\kappa$ and $\text{Ric}^{(2)}_h \leq -C_1 h + C_2 (f^{-1})^* g$ for some constants κ, C_1, C_2, with $\kappa \geq 0$. Then,

$$\Delta_h |\partial f|^2 \geq |\nabla \partial f|^2 + C_1 |\partial f|^2 - nC_2 - \kappa$$

where $n = \dim_{\mathbb{C}}(M)$.

Proof. By Theorem 2.3.1, we have

$$\Delta_h |\partial f|^2 = |\nabla \partial f|^2 - \left(h^{\gamma\overline{\delta}} R^h_{\gamma\overline{\delta}\alpha\overline{\beta}} \right) g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta} + h^{\gamma\overline{\delta}} g^{i\overline{q}} g^{p\overline{j}} R^g_{k\overline{\ell}p\overline{q}} h_{\alpha\overline{\beta}} f_i^\alpha \overline{f_j^\beta} (f^{-1})^k_\gamma \overline{(f^{-1})^\ell_\delta}.$$

We consider the second term, recalling that $h^{\gamma\overline{\delta}} R^h_{\gamma\overline{\delta}\alpha\overline{\beta}} = \text{Ric}^{(2)}_{\alpha\overline{\beta}}$. By our curvature assumptions, we have that

$$-g^{i\overline{j}} \text{Ric}^{(2)}_{\alpha\overline{\beta}} f_i^\alpha \overline{f_j^\beta} \geq C_1 g^{i\overline{j}} h_{\alpha\overline{\beta}} f_i^\alpha \overline{f_j^\beta} - C_2 g_{p\overline{q}} (f^{-1})^p_\alpha \overline{(f^{-1})^q_\beta} g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta} = C_1 |\partial f|^2 - C_2 n$$

We now analyse the last term. Choose coordinates such that $g_{i\bar{j}} = \delta^i_j, h_{\alpha\bar{\beta}} = \delta^\alpha_\beta$ and $f^\alpha_i = \lambda_i \delta^\alpha_i$, where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r \geq \lambda_{r+1} \geq 0$, where r is the rank of $\partial f = (f^\alpha_i)$. Then,

$$\delta^\gamma_\delta \delta^i_q \delta^p_j R^g_{k\bar{\ell}p\bar{q}} \delta^\alpha_\beta \lambda_i \delta^\alpha_i \lambda_j \delta^b_j \lambda^{-1}_\gamma \delta^k_\gamma \lambda^{-1}_\delta \delta^\ell_\delta = \sum_{i,k} R_{k\bar{k}i\bar{i}} \lambda^2_i \lambda^{-2}_k.$$

By the curvature assumption $\mathrm{SBC}_g \geq -\kappa$, we have that $R_{k\bar{k}i\bar{i}} \lambda^2_i \lambda^{-2}_k \geq -\kappa$. Combining all results above gives the inequality

$$\Delta_h |\partial f|^2 \geq |\nabla \partial f|^2 + C_1 |\partial f|^2 - C_2 n - \kappa.$$

□

Corollary 2.3.5 ([BP23, Theorem 1.5])**.** Let (X, g) be a compact Hermitian manifold with $\mathrm{SBC}_g \geq 0$. If $\mathrm{Ric}^{(2)}_h = 0$ for some Hermitian metric h on X, then h has the same Chern connection as g. If there is a point where $\mathrm{SBC}_g > 0$, then there are no metrics on X with vanishing second Chern Ricci curvature.

Proof. Let $f : (X, g) \to (X, h)$ denote a holomorphic map which is biholomorphic onto its image. We first assume that $\mathrm{SBC}_g \geq 0$. The Aubin–Yau inequality then yields

$$\Delta_h |\partial f|^2 \geq |\nabla \partial f|^2.$$

Since X is compact, applying the maximum principle then obtains $|\nabla \partial f|^2 = 0$. Setting $f = \mathrm{id}$ then implies that they have the same Chern connections. Moreover, if $\mathrm{SBC}_g > 0$ at a point, we get a contradiction similar to that of Theorem 2.2.8. □

Corollary 2.3.6 ([Bro22b, Corollary 4.2])**.** Let X be a compact Hermitian manifold supporting Hermitian metrics g and h. Assume $\mathrm{SBC}_g \geq -\kappa$ and $\mathrm{Ric}^{(2)}_h \leq -C_1 h + C_2 g$ for constants κ, C_1, C_2, with $\kappa \geq 0$ and $C_1 > 0$. If $\kappa \leq -C_2$, then the automorphism group $\mathrm{Aut}(X)$ is trivial.

Proof. We apply the Aubin–Yau inequality to a biholomorphic map $f : (X, g) \to (X, h)$. By the assumptions of the corollary, we have that

$$\Delta_h |\partial f|^2 \geq C_1 |\partial f|^2 + C_2 (n - 1).$$

We note that the right hand side of the inequality is always non-negative, since $n \geq 1$. Applying the maximum principle then yields a contradiction. Thus, $\mathrm{Aut}(X)$ is trivial. □

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