Curvature and Moduli – Some Intimations and Propaganda

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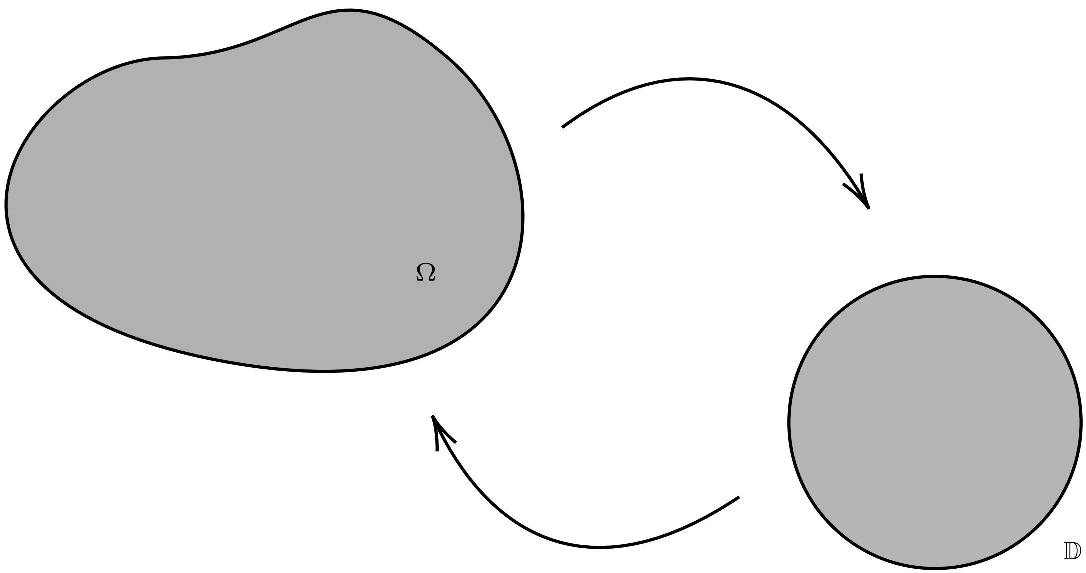
Riemann Mapping Theorem

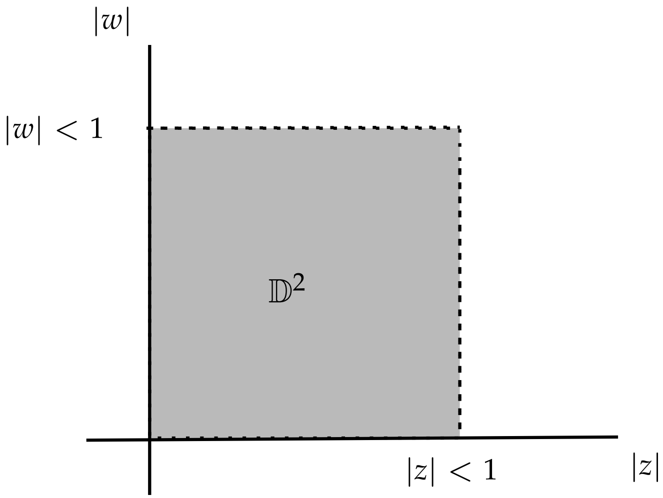
Theorem. A simply connected domain $\Omega \subsetneq \mathbb{C}$ is biholomorphic to the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.



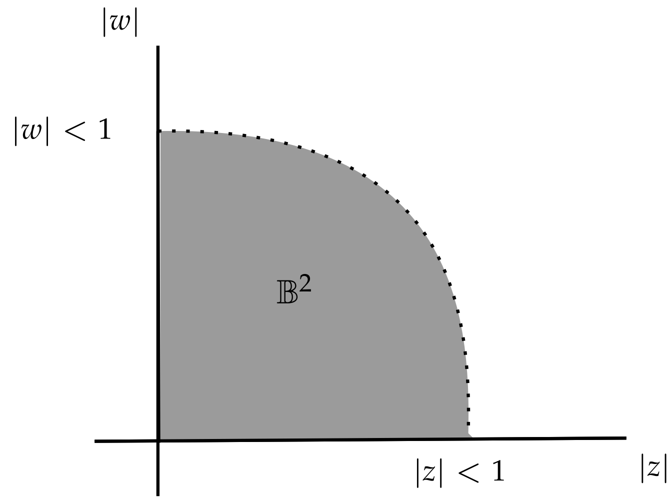
A domain is always understood to mean a connected open set in $\mathbb{C}^n$ for some $n \in \mathbb{N}$.

The Birth of Several Complex Variables

Theorem. (Poincaré). The ball $\mathbb{B}^2 := \{|z|^2 + |w|^2 < 1\}$ is not biholomorphic to the bidisk $\mathbb{D}^2 := \{|z| < 1, |w| < 1\}$.



Bidisk

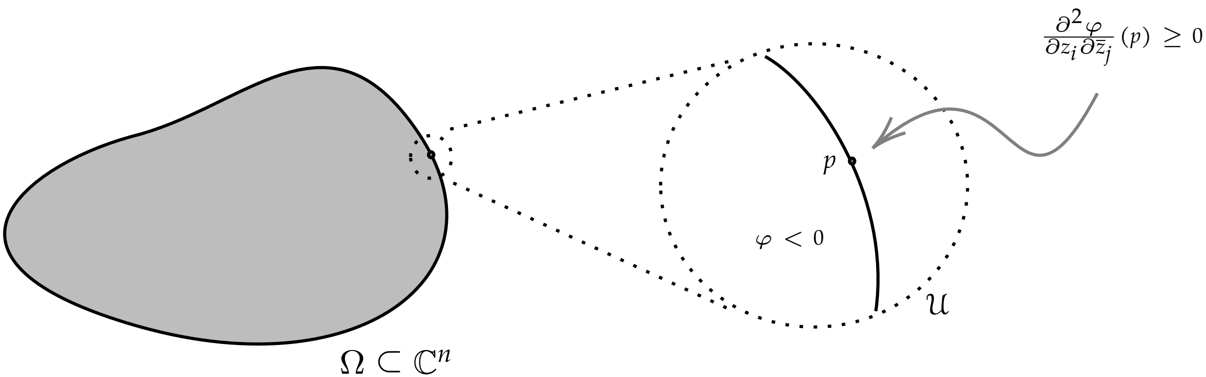


The Ball

Declare a bounded domain $\Omega \subseteq \mathbb{C}^n$ is **pseudoconvex** if for all $p \in \partial\Omega$, there is a smooth function φ defined in a neighborhood $\mathcal{U} \subset \mathbb{C}^n$ of p such that the complex Hessian[1]

$$\sqrt{-1}\partial\bar{\partial}\varphi = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right)$$

is positive semi-definite.



If $\sqrt{-1}\partial\bar{\partial}\varphi$ is positive definite, we say that Ω is **strongly pseudoconvex**.

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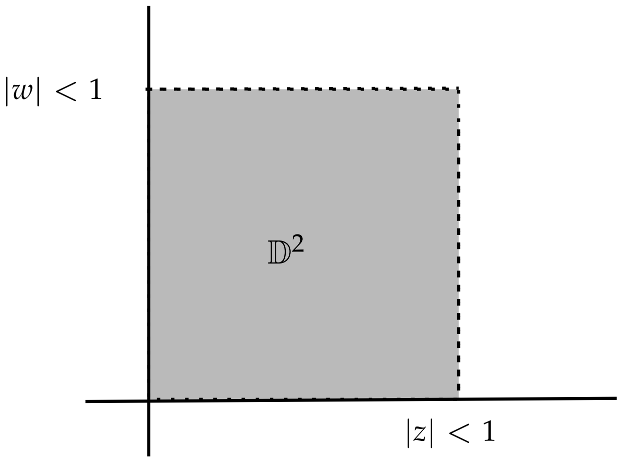
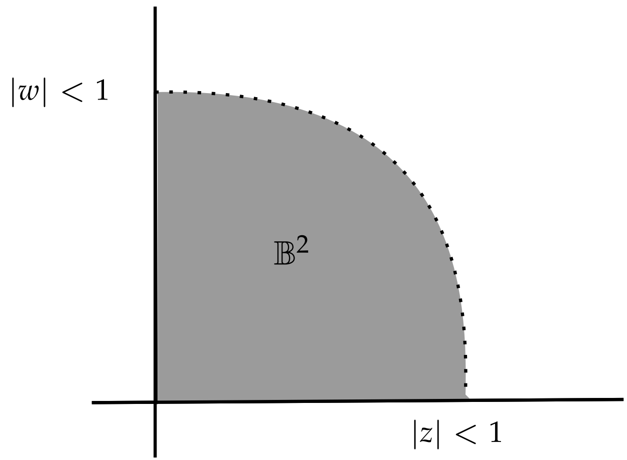
[1] The complex Hessian is the smallest refinement on the familiar Hessian such that it remains invariant under a holomorphic change of coordinates.

Pseudoconvexity and Strong Pseudoconvexity is preserved under **biholomorphism** (if the boundaries are $\mathcal{C}^\infty$–smooth).

The bidisk $\mathbb{D}^2$ is **pseudoconvex** while the ball $\mathbb{B}^2$ is **strongly pseudoconvex**.



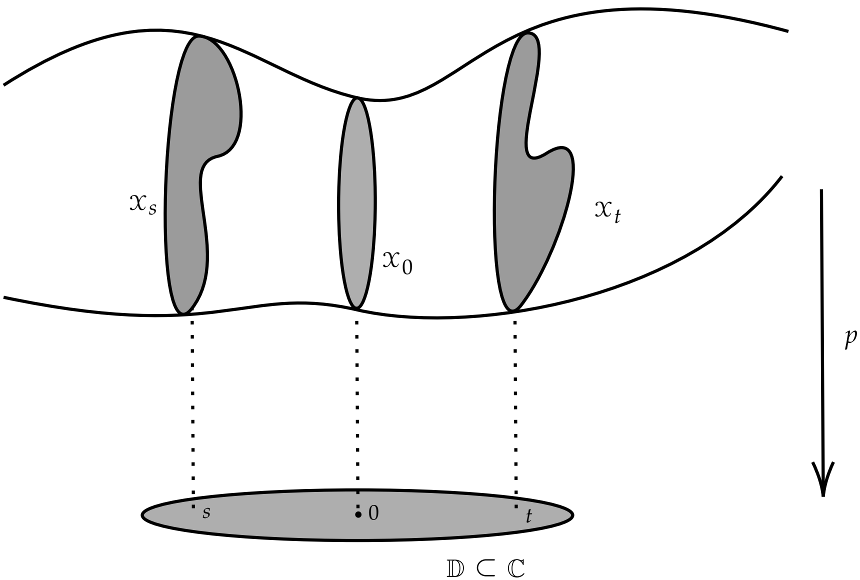
Pseudoconvex



Strongly Pseudoconvex

This discrepancy has an important consequence in terms of the behavior of disk fibrations:

A surjective holomorphic submersion $p : \mathcal{X} \to \mathbb{D}$ is said to be a **disk fibration** if every fiber $\mathcal{X}_t := p^{-1}(t)$, for $t \in \mathbb{D}$, is biholomorphic to a disk.



The projection onto one of the factors defines a disk fibration structure on both $\mathbb{D}^2$ and $\mathbb{B}^2$.

For the bidisk $\mathbb{D}^2$, the disk fibration $p : \mathbb{D}^2 \to \mathbb{D}$ is holomorphically trivial.

We say that a disk fibration $p : \mathcal{X} \to \mathcal{S}$ is locally (holomorphically) trivial if for each point $s \in \mathcal{S}$, there is an open neighborhood $\mathcal{U} \ni s$ such that

$$p^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{D}.$$

Of course, if $\mathcal{X} = \mathbb{D}^2$, for any point $s \in \mathbb{D}$, we can take $\mathcal{U} = \mathbb{D}$.

On the other hand, the disk fibration $p : \mathbb{B}^2 \to \mathbb{D}$ cannot be holomorphically trivial:

An old theorem of Royden tells us that a disk fibration is locally holomorphically trivial if and only if it is holomorphically trivial.

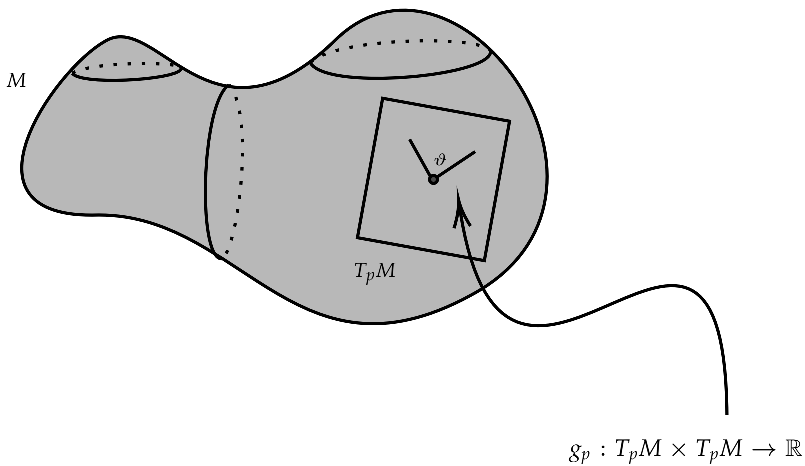
Hence, if $p : \mathbb{B}^2 \to \mathbb{D}$ is locally trivial, then $\mathbb{B}^2$ would be biholomorphic to $\mathbb{D}^2$.

The bidisk $\mathbb{D}^2$ and the ball $\mathbb{B}^2$, therefore, occupy two opposing ends from the perspective of moduli and deformation theory.

Understanding the behavior of complex manifolds in families can be difficult, and we would like to have a robust mechanism for measuring the existence or non-existence of holomorphic variation in the fibers.

Question. Can the behavior of the disk fibrations $p : \mathcal{X} \to \mathbb{D}$ be detected by looking at the curvature of metrics which reside on $\mathcal{X}$?

A Riemannian metric g on smooth manifold M is a positive definite quadratic form $g_p : T_pM \times T_pM \to \mathbb{R}$ on each of the tangent spaces T_pM such that the map $p \mapsto g_p$ is smooth.



If $(x_1, \ldots, x_n)$ are local coordinates near $p \in M$, we write $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$ for the metric in these coordinates, where $g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ are the components of the metric in these coordinates.

The Riemannian metric permits us to compute the lengths of tangent vectors.

Given a smooth curve $\gamma : [0,1] \to M$, by integrating the norms of the tangent vectors $\dot{\gamma}(t)$, we can compute its length

$$\mathrm{Length}_g(\gamma) := \int_0^1 |\dot{\gamma}(t)|_{g(t)} dt = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

This in turn gives us a distance function

$$\mathrm{dist}_g : M \times M \to \mathbb{R}$$

on M given by declaring the distance between two points $p, q \in M$ to be the infimum of the lengths of curves γ with $\gamma(0) = p$ and $\gamma(1) = q$.

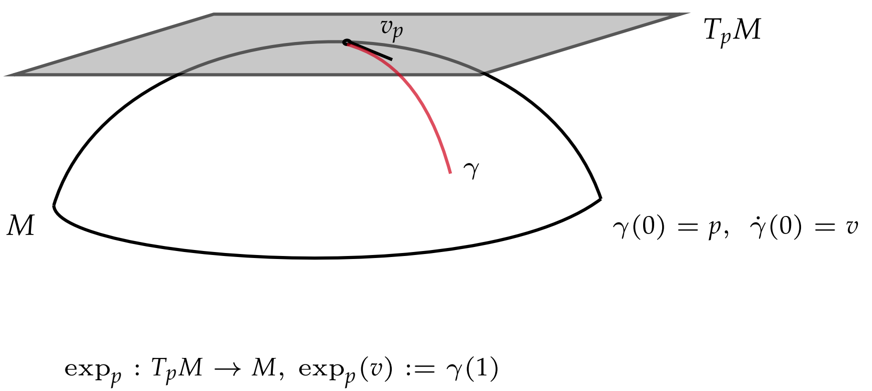
We will say that a Riemannian metric is complete if the distance function dist_g is Cauchy complete.

The curves which (locally) realize the shortest distance between points are called **geodesics**.

Define the exponential map

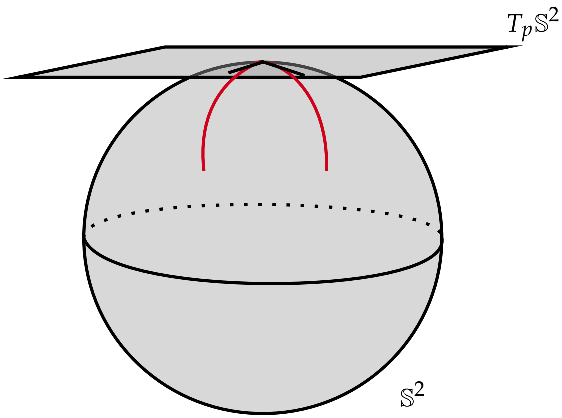
$$\exp_p : T_pM \to M, \qquad T_pM \ni v \mapsto \gamma(1) \in M,$$

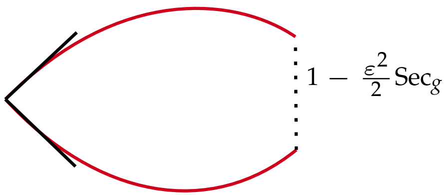
where $\gamma(1)$ denotes the endpoint of the unique geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.



The Riemannian curvature tensor $R = R_{ikj\ell}$ measures the failure of the exponential map to be an isometry:

$$g_{ij} \quad = \quad \delta_{ij} - \frac{1}{3} R_{ikj\ell} x^k x^\ell + O(|x|^3)$$





The Riemannian curvature tensor is determined by the sectional curvature $\mathrm{Sec}_g : \mathrm{Gr}_2(TM) \to \mathbb{R}$, $\mathrm{Sec}_g(u, v) := R(u, v, v, u) / |u|^2 |v|^2$, and we thus use the terms interchangably.

Examples:

- The sphere $\mathbb{S}^n$ has a metric of positive sectional curvature.
- (Wilking). There is a metric of positive sectional curvature almost everywhere on $\mathbb{S}^2 \times \mathbb{S}^2$.
- The torus has a metric of vanishing curvature.
- The ball $\mathbb{B}^n \subset \mathbb{C}^n$ has a metric of negative sectional curvature.

Riemannian manifolds with negative sectional curvature:

Theorem. (Cartan–Hadamard). A complete Riemannian manifold (M, g) with $\mathrm{Sec}_g \leq 0$ has universal cover diffeomorphic to $\mathbb{R}^n$.

In particular, the homotopy-type of $M \in (\mathrm{Sec} \leq 0)$ is localized in the fundamental group $\pi_1(M)$.

Reminder: A Riemannian manifold (M, g) is said to be complete if the distance function $\mathrm{dist}_g : M \times M \to \mathbb{R}$ (given by infimum of lengths of curves) is Cauchy complete.

Riemannian manifolds with negative sectional curvature:

Theorem. (Preissman). Let (M, g) be a compact Riemannian manifold with $\mathrm{Sec}_g < 0$. Then any abelian subgroup of the fundamental group $\pi_1(M)$ is cyclic.

In particular, compact product manifolds cannot admit metrics with $\mathrm{Sec}_g < 0$, since the fundamental group would then contain $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup.

Without compactness, negative sectional curvature is not obstructed on products:

Theorem. (Anderson). Let $f : \mathcal{E} \to \mathcal{B}$ be a smooth vector bundle over a compact Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$ with $\mathrm{Sec}_{g_{\mathcal{B}}} < 0$. Then $\mathcal{E}$ admits a complete Riemannian metric $g_{\mathcal{E}}$ with

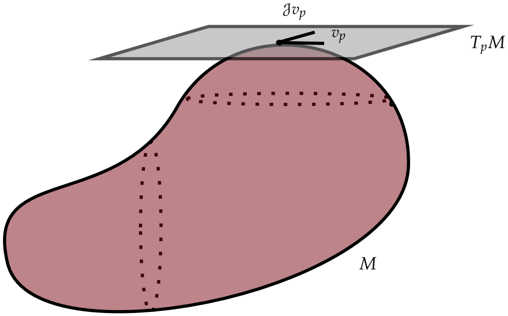
$$-a \leq \mathrm{Sec}_{g_{\mathcal{E}}} \leq -1.$$

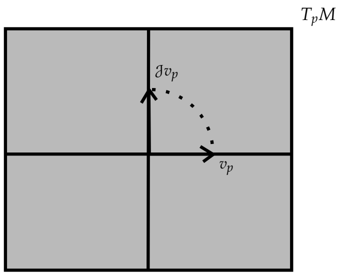
The constant $a \geq 1$ depends only on the geometry of $\mathcal{B}$ and the topology of $f : \mathcal{E} \to \mathcal{B}$.

Complex Structures

An **almost complex structure** $\mathcal{J}$ on a smooth manifold M is an endomorphism

$$\mathcal{J} : TX \to TX, \qquad \mathcal{J}^2 = -\mathrm{id}.$$



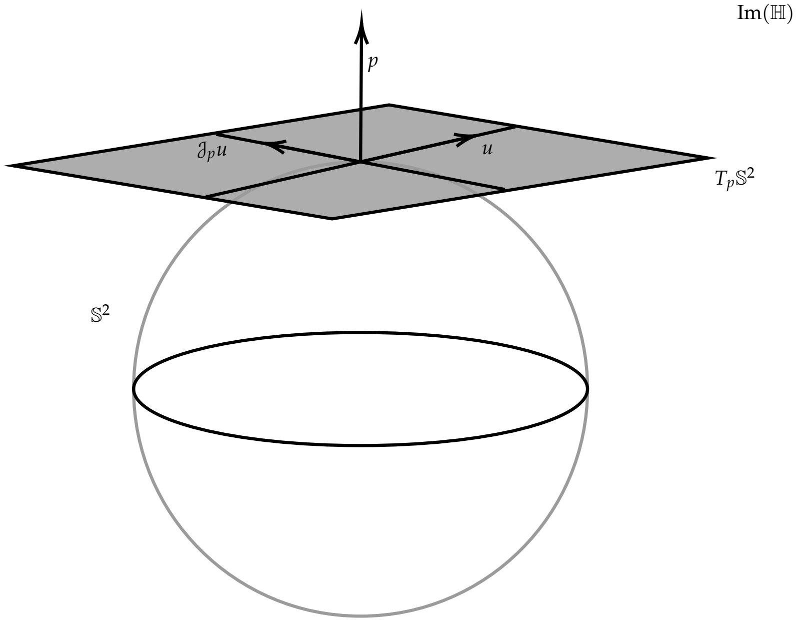


An Almost Complex Structure on $\mathbb{S}^2$.

Identify $\mathbb{S}^2 \subset \mathbb{R}^3$ with the space of unit imaginary quaternions $\mathrm{Im}(\mathbb{H}^3) \simeq \mathbb{R}^3$.

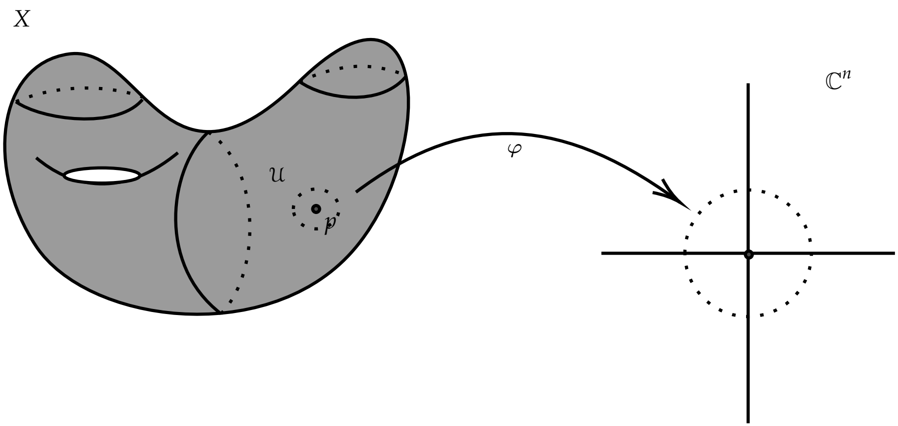
For each point $p \in \mathbb{S}^2$, we get a map $\mathcal{J}_p : T_p\mathbb{S}^2 \to T_p\mathbb{S}^2$ satisfying $\mathcal{J}_p^2 = -\mathrm{id}_{T_p\mathbb{S}^2}$, given by

$$\mathcal{J}_p(v) := p \times v.$$



In general, an **almost complex structure** $\mathcal{J} \in \mathrm{End}(TX)$ is **not sufficient** to yield **local holomorphic coordinates**.

There is an obvious **obstruction**: Suppose X is a complex manifold with holomorphic coordinates $(z_1, \ldots, z_n)$ centered at a point $p \in X$.



The tangent space to X at the point p is the complex vector space:

$$T_pX \;=\; \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \right\}.$$

Let M be a smooth manifold with almost complex structure $\mathcal{J}$.

The condition $\mathcal{J}^2 = -\mathrm{id}$ gives an eigenspace splitting

$$T_p^{\mathbb{C}} M \simeq T_p^{1,0} M \oplus T_p^{0,1} M,$$

corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

If $(x_1, \dots, x_{2n})$ are smooth coordinates on M, then $T_p^{1,0} M$ is spanned by

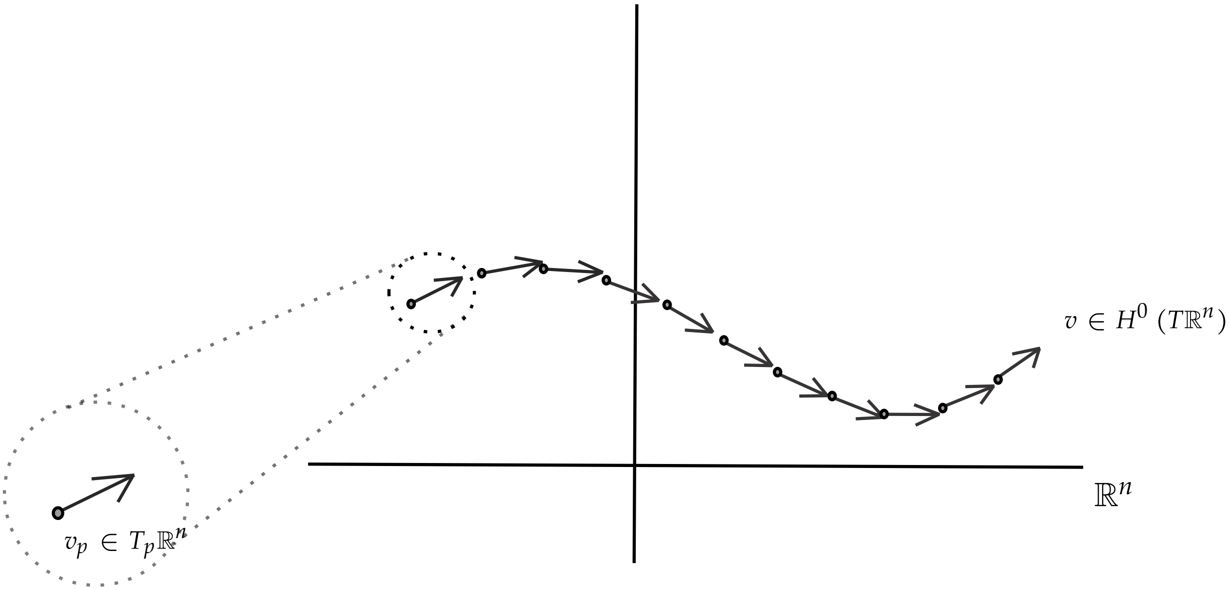
$$\frac{\partial}{\partial z_i} := \frac{\partial}{\partial x_i} - \sqrt{-1}\mathcal{J}\frac{\partial}{\partial x_i},$$

and $T_p^{0,1} M$ is spanned by

$$\frac{\partial}{\partial \bar{z}_i} := \frac{\partial}{\partial x_i} + \sqrt{-1}\mathcal{J}\frac{\partial}{\partial x_i}.$$

Hence, if an almost complex structure $\mathcal{J}$ gives rise to a system of local holomorphic coordinates, we need to be able to find a complex manifold X such that the tangent bundle of X is prescrisely $T^{1,0} M$.

We have seen this before in the context of vector fields and integral curves:



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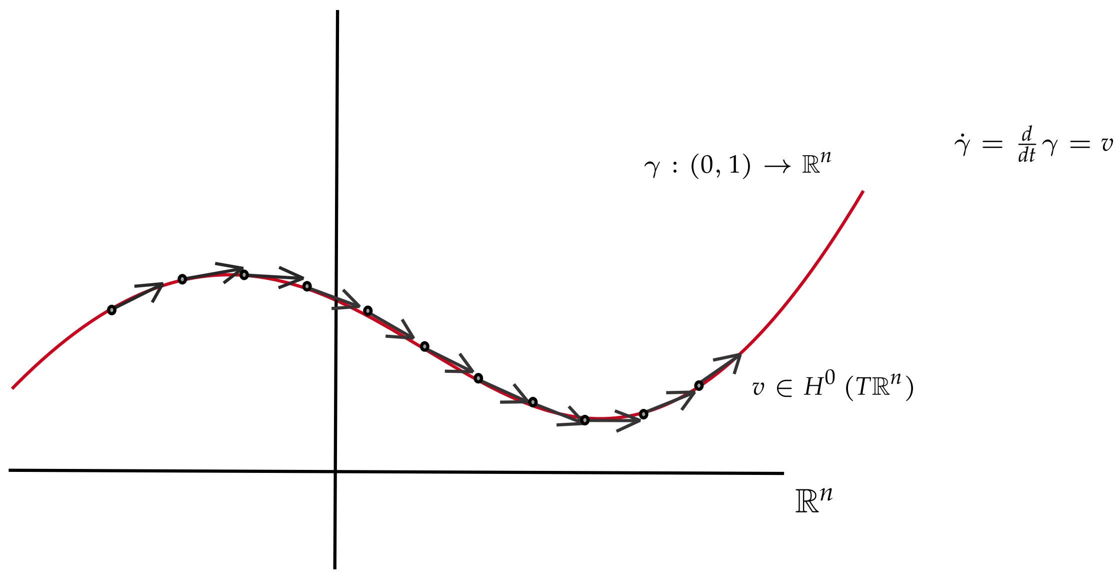
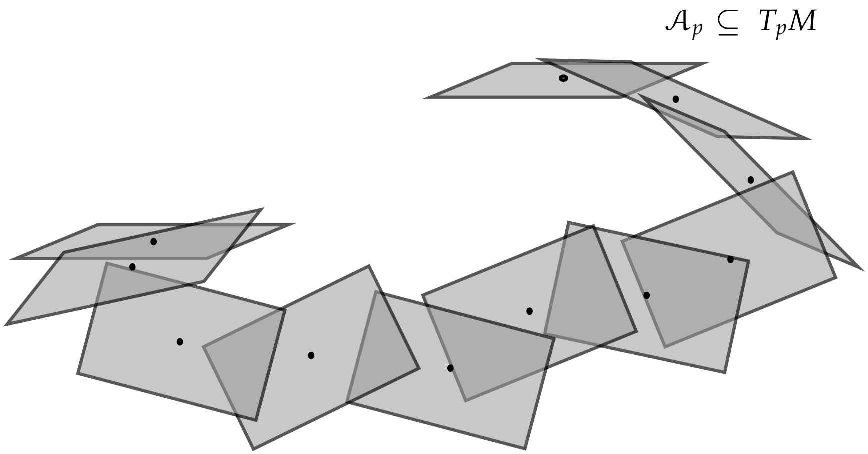


The integrability condition on the complex structure is merely a higher-dimensional version of this:



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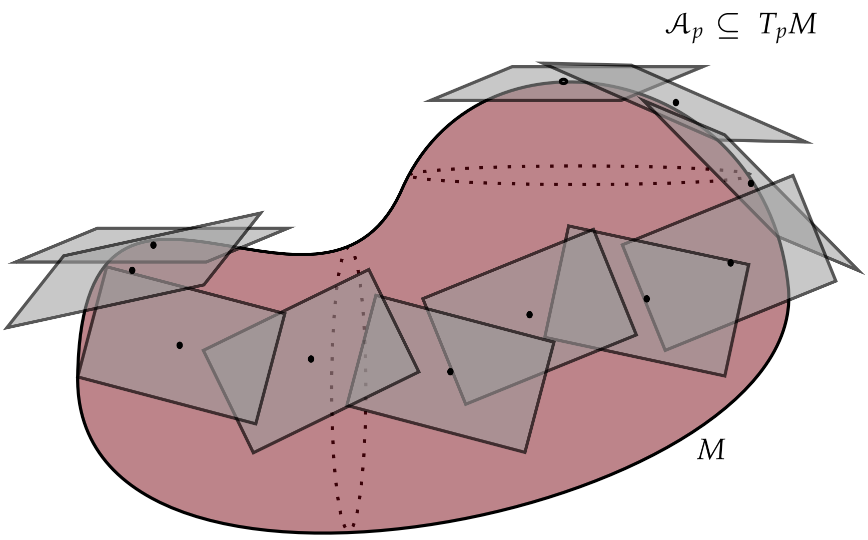

The Frobenius theorem tells us that $T^{1,0}M$ is an integrable subbundle if and only if it is closed under Lie bracket:

$$[u, v] \subseteq T^{1,0}M, \qquad \forall u, v \in T^{1,0}M.$$

This manifests as the vanishing of the Nijenhuis tensor:

$$\mathcal{N}^{\mathcal{J}}(u_0, v_0) \quad := \quad [u_0, v_0] + J([Ju_0, v_0] + [u_0, Jv_0]) - [Ju_0, Jv_0].$$

Theorem. (Newlander–Nirenberg). An almost complex structure $\mathcal{J}$ is integrable if and only if $\mathcal{N}^{\mathcal{J}} \equiv 0$.

We can repeat the almost complex structure construction on $\mathbb{S}^2$ with $\mathbb{S}^6$ – identify $\mathbb{S}^6$ with the space of unit imaginary octonions $\mathrm{Im}(\mathbb{O})$. This endows $\mathbb{S}^6$ with an almost complex structure.

If one computes the Nijenhuis tensor of this almost complex structure, however, it does not vanish precisely because the octonions are not associative.

Hermitian and Kähler Metrics

A Riemannian metric g on a complex manifold $(X, \mathcal{J})$ is said to be **Hermitian** if

$$g(\mathcal{J}u, \mathcal{J}v) \;=\; g(u, v), \qquad\qquad u, v \in TX.$$

Every complex manifold supports a Hermitian metric: Take any Riemannian metric g and set

$$h(u, v) := g(u, v) + g(\mathcal{J}u, \mathcal{J}v).$$

We say that a Hermitian metric g is **Kähler** if the 2–form

$$\omega_g(u, v) := g(\mathcal{J}u, v)$$

is **closed**.

Some examples of Kähler manifolds

- † Complex projective space $\mathbb{P}^n$ endowed with the Fubini–Study metric.
  - $\rightsquigarrow$ Projective manifolds.
- † Euclidean space $\mathbb{C}^n$ endowed with the Euclidean metric.
  - $\rightsquigarrow$ Stein manifolds (in particular, pseudoconvex domains).
- † A compact complex surface is Kähler if and only if the first Betti number is even.
  - $\rightsquigarrow$ Hopf surface $\mathbb{S}^1 \times \mathbb{S}^3$ is not Kähler.
- † The Weil–Petersson metric on the Riemann moduli space $\mathcal{M}_g$.

Holomorphic Bisectional Curvature

Let (X, ω) be a Kähler manifold. The **holomorphic bisectional curvature** is given by

$$\text{HBC}_\omega(u, v) \quad := \quad \frac{1}{|u|^2_\omega |v|^2_\omega} R(u, \mathcal{J}u, v, \mathcal{J}v),$$

where $u, v \in T^{1,0}X$.

The terminology comes from the observation that the HBC is a **sum of two sectional curvatures**:

$$\text{HBC}_\omega(u, v) \;=\; R(v_0, u_0, u_0, v_0) + R(\mathcal{J}u_0, v_0, v_0, \mathcal{J}u_0),$$

where $u = u_0 - \sqrt{-1}\mathcal{J}u_0$ and $v = v_0 - \sqrt{-1}\mathcal{J}v_0$.

The most famous result concerning the holomorphic bisectional curvature is the **Mori** and **Siu–Yau** solution of the **Frankel conjecture**:

Theorem. (Mori, Siu–Yau). Let (X, ω) be a compact Kähler manifold with $\mathrm{HBC}_\omega > 0$. Then X is biholomorphic to $\mathbb{P}^n$.

In contrast to the sectional curvature, there are compact simply connected Kähler manifolds with $\mathrm{HBC}_\omega < 0$. There were recently constructed by Mohsen.

Reminder: Structure theorems for Riemannian manifolds with Sec < 0.

Cartan–Hadamard:

$$M \in (\text{Sec} \leq 0) \implies \widetilde{M} \simeq_{\text{diffeo}} \mathbb{R}^n.$$

Preissman:

$$M \in (\text{Sec} < 0) \cap (\text{Cmpct}) \implies M \not\simeq M_1 \times M_2.$$

Anderson:

$$\mathcal{B} \in (\text{Sec} < 0) \cap (\text{Cmpct}) \implies \text{Vect}_{\mathcal{C}^\infty}(\mathcal{B}) \subseteq (-a \leq \text{Sec} \leq -1).$$

The Complex-Analytic Category:

Replace:

- smooth vector bundles by holomorphic vector bundles $f : \mathcal{E} \to \mathcal{B}$
- sectional curvature by the holomorphic bisectional curvature.

Question. Let $f : \mathcal{E} \to \mathcal{B}$ be a holomorphic vector bundle, where $\mathcal{B}$ is compact and admits a Hermitian metric ω with ${}^c\text{HBC}_\omega < 0$. Does $\mathcal{E}$ admit a complete Hermitian metric with $-a \leq {}^c\text{HBC} \leq -1$, for some constant $a > 1$?

The answer turns out to be **false**, by a result of F. Zheng:

Theorem. (Zheng). Let $\mathcal{X} := X \times Y$ be a product complex manifold with X compact. Then $\mathcal{X}$ does not admit a Hermitian metric ω with

$$^{c}\mathrm{HBC}_{\omega} \leq -1.$$

In fact, Zheng's theorem asserts that $\mathcal{X}$ does not even admit a (possibly non-complete) Hermitian metric with $^{c}\mathrm{HBC}_{\omega} \leq -1$.

A Theorem of Paul Yang

Theorem. (Yang). Let $\mathcal{F} \hookrightarrow \mathcal{X} \to \mathcal{B}$ be a holomorphic fiber bundle with $\mathcal{F}$ compact. Then $\mathcal{X}$ does not admit a complete Kähler metric with $\mathrm{HBC}_{\omega} \leq -\kappa_0 < 0$.

The following theorem of Fischer and Grauert shows that holomorphic fiber bundles with compact fiber are trivial in the following sense:

Theorem. (Fischer–Grauert). Let $p : \mathcal{X} \to \mathcal{S}$ be a holomorphic family of compact complex manifolds. The fibers of p are all biholomorphic if and only if p is a holomorphic fiber bundle.

A Theorem of Paul Yang

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Corollary. Let $p : \mathcal{X} \to \mathcal{B}$ be a holomorphic family of compact complex manifolds. If $\mathcal{X}$ admits a complete Kähler metric with $\mathrm{HBC}_\omega \leq -\kappa_0 < 0$, there must be **non-trivial** holomorphic **variation** in the fibers.

The bisectional curvature must be **bounded away from zero**:

Theorem. (Klembeck). There is a complete **Kähler** metric on $\mathbb{C}^n$ with

$$\text{HBC}_\omega > 0.$$

Seshadri gave a small modification of Klembeck's construction, showing:

Theorem. (Seshadri = Klembeck+ε). There is a complete **Kähler** metric on $\mathbb{C}^n$ with

$$\text{HBC}_\omega < 0.$$

The narrative thus far:

- The bidisk $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D} \subseteq \mathbb{C}^2$ is a holomorphically trivial disk fibration.
- The ball $\mathbb{B}^2$ is a disk fibration which cannot be locally trivial.
- In the Riemannian category, Preissman's theorem ensures that compact manifolds with negative sectional curvature cannot be trivial bundles.
- Zheng: Product manifolds with one of the factors being compact do not admit Hermitian metrics with HBC ≤ -1.
- Yang: Holomorphic fiber bundles (holomorphic families with all fibers biholomorphic) with compact fiber do not admit metrics with HBC ≤ -1.
- Klembeck, Seshadri – The curvature must be bounded away from zero.

Curvature of the product metric on the bidisk $\mathbb{D}^2$:

(†) $\mathrm{Sec}(\mathbb{D}^2) \leq 0$.

(†) $\mathrm{HBC}(\mathbb{D}^2) \leq 0$.

Curvature of the Poincaré metric on the ball $\mathbb{B}^2$:

(†) $-4 \leq \mathrm{Sec}(\mathbb{B}^2) \leq -1$.

(†) $-2 \leq \mathrm{HBC}(\mathbb{B}^2) \leq -1$.

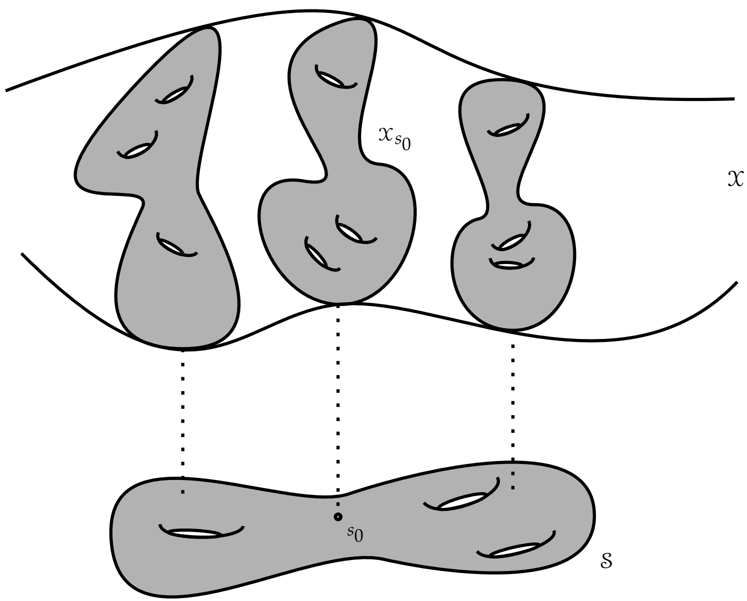
Recall that $p : \mathbb{D}^2 \longrightarrow \mathbb{D}$ is a trivial disk fibration, while $p : \mathbb{B}^2 \longrightarrow \mathbb{D}$ is a necessarily non-trivial disk fibration.

The Conjectural Picture:

Conjecture. Let $f : \mathcal{X} \to \mathcal{S}$ be a holomorphic family of complex manifolds. Suppose $\mathcal{X}$ admits a **complete** Hermitian metric with HBC $\leq -\kappa_0 < 0$. Then f is **not** (holomorphically) **locally trivial**.

Kodaira Fibration Surfaces

Let $p : \mathcal{X} \to \mathcal{S}$ be a surjective holomorphic submersion onto a compact Riemann surface of genus $b \geq 2$ with fibers being compact Riemann surfaces of genus $g \geq 2$. If there fibers are **not all biholomorphic**, then we say that $p : \mathcal{X} \to \mathcal{S}$ is a **Kodaira Fibration Surface**.



Curvature of the Total Space of Kodaira Fibrations

Theorem. (To–Yeung) Let $p : \mathcal{X} \to \mathcal{S}$ be a Kodaira fibration surface. Then $\mathcal{X}$ admits a Kähler metric with $\text{HBC}_\omega < 0$.

The structure of the argument is just as important as the result:

- The fibers of a KFS are Riemann surfaces of genus $g \geq 2$. So we get a moduli map $\mu : \mathcal{S} \to \mathcal{M}_g$ into the moduli space of genus $g \geq 2$ Riemann surfaces.
- Define a map $\tau : \mathcal{X} \to \mathcal{M}_{g,1}$ by sending $x \in \mathcal{X}$ to the biholomorphism class of the marked Riemann surface $\mathcal{X}_{p(x)} - \{x\}$, where $\mathcal{X}_{p(x)} := p^{-1}(p(x))$ is the fiber over $p(x)$.
- The Weil–Petersson metric ω_{WP} on $\mathcal{M}_{g,1}$ has strictly negative bisectional curvature. Thus, we obtain a metric on $\mathcal{X}$ by pulling back the Weil–Petersson metric from $\mathcal{M}_{g,1}$ to $\mathcal{X}$.

KFS = Kodaira fibration surface = the total space of non-trivial family of genus $\geq$ 2 Riemann surfaces over a genus $\geq$ 2 Riemann surface.

Question. (Mok). Does the bidisk $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D}$ admit a complete Kähler metric with $\mathrm{HBC}_\omega \leq -\kappa_0 < 0$?

Thanks for listening!