EXAMPLES OF UNBOUNDED DOMAINS THAT ARE KOBAYASHI HYPERBOLIC

KYLE BRODER

Every bounded domain $\Omega \subset \mathbf{C}^n$ is Brody hyperbolic in the sense that every holomorphic map $\mathbf{C} \to \Omega$ is constant. This is an elementary consequence of Liouville's theorem. Brody hyperbolicity is strictly weaker than Kobayashi hyperbolicity (defined by the Kobayashi pseudodistance being non-degenerate). For instance, Barth [1] produced the following example of a pseudoconvex Brody hyperbolic domain that is not Kobayashi hyperbolic:

Example 1. Let $u : \mathbf{D} \to \mathbf{R}$ be the subharmonic function defined by

$$u(z) \; := \; \max\left(\log|z|, \sum_{k=1}^{\infty} \frac{1}{k^2} \log|z - 1/k| \right).$$

The domain

$$\Omega := \{(z,w) \in \mathbf{D} \times \mathbf{C} : |w| < \exp(-u(z))\}$$

is pseudoconvex, Brody hyperbolic, but not Kobayashi hyperbolic.

We collect some examples from [4] of unbounded Kobayashi hyperbolic domains.

Theorem 2. (Gaussier–Shcherbina). Let $\Omega \subseteq \mathbf{C}^n$ be an unbounded domain. If there is a bounded non-vanishing holomorphic function on Ω that vanishes as $\|z\| \to +\infty$, then Ω is Kobayashi hyperbolic.

Example 3. Let $\Omega_\varepsilon := \{(z,w) \in \mathbf{C}^2 : |w| < |z|, \text{ and } |z| > \varepsilon\}$. For all $\varepsilon > 0$, the domain Ω_ε is Kobayashi hyperbolic. But Ω_0 contains a copy of $\mathbf{C}^\star \simeq \{(z,w) \in \mathbf{C}^2 : w = 0 \text{ and } |z| > 0\}$, and is thus not Kobayashi hyperbolic.

Definition 4. Let $\Omega \subseteq \mathbf{C}^n$ be an unbounded domain. The *core of* Ω is defined to be the set of points $p \in \Omega$ such that every bounded continuous plurisubharmonic function on Ω fails to be smooth and strictly plurisubharmonic near p.

Remark 5. Since the function $z \mapsto \|z\|^2$ is strictly plurisubharmonic on $\mathbf{C}^n$, every bounded domain in $\mathbf{C}^n$ has an empty core. It follows from the biholomorphic invariance of the core that an unbounded domain with a non-empty core does not admit a bounded representation.

Example 6. Harz–Shcherbina–Tomassini [3, Theorem 1.2] constructed for every $n \geq 2$, an unbounded strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$ with smooth boundary such that the core is not empty and contains no analytic subvariety of positive dimension.

Example 7. Shcherbina–Zhang [6] constructed a strictly pseudoconvex domain in $\mathbf{C}^2$ with smooth boundary and non-empty core that is Kobayashi and Bergman complete, but has no non-constant holomorphic functions.

Example 8. Green's seminal work [2] contains an example of a Kobayashi hyperbolic domain that admits no bounded representation. Indeed, if Ω is the complement of $(2n+1)$ hyperplanes in general position in $\mathbf{P}^n$, then Green showed that Ω is Kobayashi hyperbolic. We may assume that one of these planes is the hyperplane at infinity, so Ω is contained in $\mathbf{C}^n$. Suppose Ω admits a biholomorphic Φ onto a bounded domain $\mathcal{D}$. Then the hyperplanes would be removable singularities for Φ and Φ would be bounded in a neighborhood of the hyperplanes. Hence, Φ extends to a holomorphic map on $\mathbf{C}^n$, violating Liouville's theorem.

Definition 9. A domain $\Omega \subseteq \mathbf{C}^n$ is *rigid* if there is a function Ψ in $\mathbf{C}^{n-1}$ such that

$$\Omega \ = \ \{(z, \zeta) \in \mathbf{C}^{n-1} \times \mathbf{C} : \mathrm{Re}(\zeta) > \Psi(z)\}.$$

Remark 10. A rigid domain Ω is pseudoconvex if and only if the function Ψ is plurisubharmonic in $\mathbf{C}^{n-1}$.

Theorem 11. ([5]). The existence of the Kobayashi and the Bergman metrics for pseudoconvex domains in $\mathbf{C}^2$ of the form

$$\Omega_H \ := \ \{(z, \zeta) \in \mathbf{C}^2 : \mathrm{Re}(\zeta) > H(z, \mathrm{Im}(\zeta))\},$$

for H a continuous function on $\mathbf{C} \times \mathbf{R}$ is equivalent to the core of Ω_H being empty.

References

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