New Vanishing Results for Kähler Manifolds

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MATRIX Meeting 2025

Let (M^n, g) be a compact connected Riemannian n-manifold and ϕ a k-form on M.

The Weitzenböck Formula for ϕ is given by:

$$\Delta\phi = (dd^* + d^*d)\,\phi = \nabla^*\nabla\phi + \mathrm{Ric}_L(\phi).$$

Here $\mathrm{Ric}_L(\phi)$ is a contraction of $\nabla^2\phi$ which as Weitzenböck realized had to be a contraction of $R \otimes \phi$ as it is a 0th order invariant.

Bochner-Yano: When the form is harmonic, $\Delta\phi = 0$, we obtain

$$0 = g\left(\nabla^*\nabla\phi, \phi\right) + g\left(\mathrm{Ric}_L\left(\phi\right), \phi\right).$$

Using,

$$\int g\left(\nabla^*\nabla\phi, \phi\right) = \int \left|\nabla\phi\right|^2 \geq 0$$

we see that

$$g\left(\mathrm{Ric}_L\left(\phi\right), \phi\right) \geq 0 \Rightarrow \left|\nabla\phi\right|^2 = 0,\ g\left(\mathrm{Ric}_L\left(\phi\right), \phi\right) = 0.$$

When $g\left(\mathrm{Ric}_L\left(\phi\right), \phi\right) \geq 0$ on all k-forms Hodge theory shows

$$b_k = \dim H^k\left(M\right) \leq \binom{n}{k} = \dim H^k\left(S^1 \times \cdots \times S^1\right)$$

and if in addition $g\left(\mathrm{Ric}_L\left(\phi\right), \phi\right) > 0$ at a point then

$$b_k = 0.$$

The techniques involved in our results depend on the basic Lie algebra actions on tensors derived from the regular representation. For an endomorphism $S \in \mathfrak{gl}(T_xM)$ we have

$$(S\omega)(v_1, \ldots, v_k) = -\sum_{i=1}^{k} \omega(v_1, \ldots, Sv_i, \ldots, v_k).$$

With an ONB $A_\alpha \in \Lambda^2 T_xM$ of eigenvectors $R(A_\alpha) = \lambda_\alpha A_\alpha$ the curvature term in the Bochner formula becomes:

$$g(\mathrm{Ric}_L(\phi), \phi) = \sum \lambda_\alpha |A_\alpha \phi|^2.$$

This formula is due to Poor and leads immediately to a proof of the Gallot-Meyer classification of manifolds with nonnegative curvature operator.

We say that a self-adjoint operator is r-positive provided its eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ satisfy

$$\lambda_1 + \cdots + \lambda_{\lfloor r \rfloor} + (r - \lfloor r \rfloor)\,\lambda_{\lfloor r \rfloor + 1} > 0.$$

If we think of

$$g\left(\mathrm{Ric}_L\left(\phi\right), \phi\right) = \sum \lambda_\alpha \left|A_\alpha \phi\right|^2$$

as a weighted sum of the eigenvalues, then the sum of the weights

$$\sum \left|A_\alpha \phi\right|^2 = W$$

compared to the maximum value of each weight

$$\max \left|A\phi\right|^2 \leq m$$

imply that the weighted sum is non-negative provided

$$\lambda_1 + \cdots + \lambda_{\lfloor \frac{W}{m} \rfloor} + \left(\frac{W}{m} - \left\lfloor \frac{W}{m} \right\rfloor\right) \lambda_{\lfloor \frac{W}{m} \rfloor + 1} > 0.$$

This strategy carries over to Kähler manifolds (X^n, g) where X is a complex manifold of complex dimension n. Here k-forms on the complexified tangent bundle $T^{\mathbb{C}}M$ are further divided into (p, q)-forms, $p + q = k$, that with multi index notation look like

$$\phi = \sum \phi_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}.$$

The corresponding cohomology groups are denoted by $H^{p,q}(X)$. Complex conjugation shows that (p, q)-forms are conjugate isomorphic to (q, p)-forms. So it suffices to consider real forms in $H^{p,q} \oplus H^{q,p}$ if we wish to control Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$.

We consider the Kähler curvature operator

$$\mathcal{K} : \Lambda^{1,1} T^{\mathbb{C}} M \to \Lambda^{1,1} T^{\mathbb{C}} M$$

with an ONB of eigenvalues $\mathcal{K}(A_\alpha) = \lambda_\alpha A_\alpha$ and obtain a similar formula

$$g\left(\mathrm{Ric}_L(\phi), \bar{\phi}\right) = \sum \lambda_\alpha |A_\alpha \phi|^2 .$$

Bochner 1946: If Ric is $2p$-positive for $p = 1, ..., n$, then $h^{p,0} = 0$.

Ogiue-Tachibana 1978: If $\mathcal{K}$ is positive, then $h^{p,q} = 0$ for $p \neq q$ and $h^{p,p} = 1$, i.e., X^n has the cohomology of $\mathbb{P}^n$.

P-Wink 2021: If $\mathcal{K}$ is $3 - \frac{2}{n}$ positive, then X^n has the cohomology of $\mathbb{P}^n$.

This result is not ideal and in fact is much better for (p, p)-forms requiring only $n/2$-positivity. It is possible that this is due to the fact that we only use that $\mathcal{K}$ is self-adjoint thus ignoring the Bianchi identity.

Significant improvements can be made if we assume the metric is Einstein. In the Kähler case this implies that the Kähler form $\omega \in \Lambda^{1,1}$ is an eigenvector for $\mathcal{K}$: $\mathcal{K}(\omega) = \rho = \lambda\omega$, where λ is the Einstein constant. Thus $\mathcal{K}$ respects the splitting

$$\Lambda^{1,1} = \Lambda_1^{1,1} \oplus \Lambda_0^{1,1}$$

where $\Lambda_1^{1,1} = \mathrm{span}(\omega)$ and $\Lambda_0^{1,1}$ is the orthogonal complement of primitive forms, i.e., cannot be factored $\phi = \varphi \wedge \omega$.

This allows us to to find an ONB A_α for $\Lambda_0^{1,1}$ such that $\mathcal{K}(A_\alpha) = \lambda_\alpha A_\alpha$ and for a primitive ϕ:

$$\begin{aligned} g\left(\mathrm{Ric}_L(\phi), \bar{\phi}\right) &= \lambda \frac{(p-q)^2}{n} |\phi|^2 + \sum \lambda_\alpha |A_\alpha \phi|^2 \\ &= \sum \lambda_\alpha \left(\frac{(p-q)^2}{n(n-1)} |\phi|^2 + |A_\alpha \phi|^2 \right). \end{aligned}$$

If we assume $|\phi|^2 = 1$, then

$$\sum \left(\frac{(p-q)^2}{n(n-1)} |\phi|^2 + |A_\alpha \phi|^2 \right) = (n+1)(p+q) + 2pq$$

and each term can can be estimated by

$$\frac{(p-q)^2}{n(n-1)} |\phi|^2 + |A_\alpha \phi|^2 \leq \left(\frac{(p-q)^2}{n(n-1)} + p + q \right).$$

Define the ratio of the right-hand sides:

$$\Gamma_{p,q} = \frac{(n+1)(p+q)+2pq}{\frac{(p-q)^2}{n(n-1)}+p+q}$$

Broder-Nienhaus-P-Stanfield-Wink: If $\mathcal{K}$ is $\Gamma_{p,q}$ positive, then primitive harmonic (p,q)-forms vanish.

- In general: $\Gamma_{p,q} > n$ with the worst estimates for $q = 0$, but in reality for $q = 1$.
- $\Gamma_{p,p} = n + 1 + p$ which is a big improvement over the old estimate of $n + 1 - p$.

For Kähler manifolds the symmetries of the curvature tensor $R_{a\bar{b}c\bar{d}}$ however allow for a more efficient self-adjoint operator that does include the Bianchi identity. The key identities used for $\mathcal{K}$ are

$$R_{a\bar{b}c\bar{d}} = R_{c\bar{d}a\bar{b}} = -R_{\bar{b}ac\bar{d}}.$$

The Bianchi identity is encoded in the following symmetries

$$R_{a\bar{b}c\bar{d}} = R_{c\bar{b}a\bar{d}} = R_{a\bar{d}c\bar{b}}.$$

This tells us that the curvature tensor is a self-adjoint operator on the vector bundle of symmetric holomorphic tensors $S^{2,0}$ spanned by

$$\partial_a \odot \partial_b = \partial_a \otimes \partial_b + \partial_b \otimes \partial_a.$$

This operator $\mathcal{C} : S^{2,0} \to S^{2,0}$ was introduced by Calabi-Vesentini in an important paper (1960) that initiated the study of rigidity of locally symmetric spaces. We decided to name it the *Calabi curvature operator* in his memory.

C-V observed that any self-adjoint operator on $S^{2,0}$ is in fact an algebraic Kähler curvature tensor.

C-V also calculated the eigenvalues of $\mathcal{C}$ for the irreducible Hermitian symmetric spaces. In constant holomorphic curvature it is a homothety. For higher rank irreducible symmetric spaces $\mathcal{C}$ has precisely two eigenvalues of opposite sign. Among these examples the complex quadric

$$\frac{SO(2+n)}{SO(2)\times SO(n)}$$

is the most positive with the eigenvalues satisfying

$$\sigma_1 + \cdots \sigma_{\lfloor \frac{n}{2} \rfloor} + \left(\frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor \right) \sigma_{1+\lfloor \frac{n}{2} \rfloor} = 0.$$

Ogiue-Tachibana 1978: If $\mathcal{C}$ is positive, then the space has the cohomology of $\mathbb{P}^n$.

Broder-Nienhaus-P-Stanfield-Wink: If $\mathcal{C}$ is $\frac{n}{2}$-positive, then X^n has the cohomology of $\mathbb{P}^n$.

More generally, if $\mathcal{C}$ is $n/2$-nonnegative, then one of the following cases holds:

- The holonomy is irreducible and the space either has the cohomology of $\mathbb{P}^n$ or is isometric to the complex quadric.
- The holonomy is reducible and a finite cover is isometric to $T^k \times Y$, where Y is a product of spaces that are biholomorphic to projective spaces.

In the first case it follows from Berger's classification of holonomy groups that the holonomy is either $U(n)$ or the space is symmetric as it can't be Ricci flat. In the second case the reducibility introduces enough zero eigenvalues that $\mathcal{C}$ is forced to be nonnegative. In particular, the bisectional curvature is nonnegative and we can use Mok's classification (1988).

Consider an ONB of eigenvalues, $\mathcal{C}(S_\alpha) = \sigma_\alpha S_\alpha$. The holomorphic tensors S_α can be type changed to maps $T^{1,0} \to T^{0,1}$. As such they act on tensors and in particular (p,q)-forms. However, $S_\alpha : \Lambda^{p,q} \to \Lambda^{p+1,q-1}$.

Nvertheless we have

$$g\left(\mathrm{Ric}_L(\phi), \bar{\phi}\right) = 2\sum \sigma_\alpha \left|S_\alpha \phi\right|^2.$$

If ϕ is primitive and $|\phi|^2 = 1$, then the total weight is

$$\sum \left|S_\alpha \phi\right|^2 = \frac{1}{4}\left((n+1)(p+q) - 2pq\right)$$

and when $\phi \in \Lambda^{p,q} \oplus \Lambda^{q,p}$ is real, then we have three estimates for the individual weights that combine to the inequality:

$$\left|S_\alpha \phi\right|^2 \leq \frac{1}{2} + \min\left\{p, q, \frac{\sqrt{pq}}{2}\right\}.$$

If we define

$$\Upsilon_{p,q} = \frac{(n+1)(p+q) - 2pq}{2 + \min\left\{4p, 4q, 2\sqrt{pq}\right\}},$$

then real primitive harmonic forms in $\Lambda^{p,q} \oplus \Lambda^{q,p}$ vanish provided $\mathcal{C}$ is $\Upsilon_{p,q}$-positive.

Here:

- $\Upsilon_{p,q} \geq \frac{n}{2}$ and the minimum is attained when $p = q = 1$.
- $\Upsilon_{p,p} = \frac{(n+1)p - p^2}{1+p}$.
- $\Upsilon_{p,0} = \frac{(n+1)p}{2}$ and in particular $c_{n,0} = \dim S^{2,0}$.