Vanishing of Hodge Numbers for K-E Manifolds

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Let (M^n, g) be a compact connected Riemannian n-manifold and ϕ a k-form on M.

The Weitzenböck Formula:

$$\Delta\phi = (dd^* + d^*d)\,\phi = \nabla^*\nabla\phi + \mathrm{Ric}_L(\phi),$$

here $\mathrm{Ric}_L(\phi)$ is a contraction of $\nabla^2\phi$ which as Weitzenböck realized had to be a contraction of $R \otimes \phi$ as it is a 0th order invariant.

Bochner-Yano: When the form is harmonic, $\Delta\phi = 0$, we obtain

$$0 = g(\nabla^*\nabla\phi, \phi) + g(\mathrm{Ric}_L(\phi), \phi).$$

Here

$$\int g\left(\nabla^* \nabla \phi, \phi\right) = \int |\nabla \phi|^2 \geq 0.$$

In particular,

$$g\left(\mathrm{Ric}_L(\phi), \phi\right) \geq 0 \Rightarrow |\nabla \phi|^2 = 0,\ g\left(\mathrm{Ric}_L(\phi), \phi\right) = 0.$$

When $g\left(\mathrm{Ric}_L(\phi), \phi\right) \geq 0$ on all k-forms Hodge theory shows

$$b_k = \dim H^k(M) \leq \binom{n}{k} = \dim H^k\left(S^1 \times \cdots \times S^1\right)$$

and if in addition $g\left(\mathrm{Ric}_L(\phi), \phi\right) > 0$ at a point then

$$b_k = 0.$$

This strategy carries over to Kähler manifolds (X^n, g) where X is a complex manifold of complex dimension n. Here k-forms on the complexified tangent bundle $T^{\mathbb{C}}M$ are further divided into (p,q)-forms, $p+q=k$, that look like

$$\phi = \sum \phi_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}.$$

The corresponding cohomology groups are denoted by $H^{p,q}(X)$. Complex conjugation shows that (p,q)-forms are conjugate isomorphic to (q,p)-forms. So it suffices to consider real forms in $H^{p,q} \oplus H^{q,p}$ if we wish to control Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$.

The techniques involved in our results depend on the basic Lie algebra actions on tensors derived from the regular representation. For an endomorphism $S \in \mathfrak{gl}(T_x M)$ we have

$$(S\omega)(v_1, \ldots, v_p) = -\sum_{i=1}^{p} \omega(v_1, \ldots, Sv_i, \ldots, v_p).$$

With this notation we can in the Riemannian case select an ONB $A_\alpha \in \Lambda^2 T_x M$ of eigenvectors $R(A_\alpha) = \lambda_\alpha A_\alpha$ and the curvature term in the Bochner formula becomes:

$$g(\mathrm{Ric}_L(\phi), \phi) = \sum \lambda_\alpha |A_\alpha \phi|^2,$$

where A_α acts as skew-symmetric transformations on TM.

This formula is due to Poor and leads immediately to a proof of the Gallot-Meyer classification of manifolds with nonnegative curvature operator.

P-Wink used this formula to obtain restrictions on Betti and Hodge numbers with less restrictive curvature assumptions. For Kähler manifolds we consider the Kähler curvature operator

$$K : \Lambda^{1,1} T^{\mathbb{C}} M \to \Lambda^{1,1} T^{\mathbb{C}} M$$

and consider an ONB of eigenvalues $K(A_\alpha) = \lambda_\alpha A_\alpha$ and obtain a similar formula

$$g\left(\mathrm{Ric}_L(\phi), \bar{\phi}\right) = \sum \lambda_\alpha |A_\alpha \phi|^2,$$

where A_α acts as skew-Hermitian operators on $T^{\mathbb{C}} M$.

We say that a self-adjoint operator is r-positive provided its eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ satisfy

$$\lambda_1 + \cdots + \lambda_{\lfloor r \rfloor} + (r - \lfloor r \rfloor) \lambda_{\lfloor r \rfloor + 1} > 0.$$

Bochner 1946: If Ric is $2p$-positive for $p = 1, ..., n$, then $h^{p,0} = 0$.

Ogiue-Tachibana: 1978: If K is positive, then $h^{p,q} = 0$ for $p \neq q$ and $h^{p,p} = 1$, i.e., X^n has the cohomology of $\mathbb{P}^n$.

P-Wink 2021: If K is $2 + 2/n$ positive, then X^n has the cohomology of $\mathbb{P}^n$.

This result is not ideal and is only optimal for (p, p)-forms requiring only $(n + 1 - p)$-positivity. It is possible that this is due to the fact that we only use that K is self-adjoint thus ignoring the Bianchi identity.

Significant improvements can be made if we assume the metric is Einstein. In the Kähler case this implies that the Kähler form $\omega \in \Lambda^{1,1}$ is an eigenvector for K: $K(\omega) = \rho = \lambda\omega$, where λ is the Einstein constant. Thus K respects the splitting

$$\Lambda^{1,1} = \Lambda^{1,1}_1 \oplus \Lambda^{1,1}_0$$

where $\Lambda^{1,1}_1 = \mathrm{span}(\omega)$ and $\Lambda^{1,1}_0$ is the orthogonal complement of primitive forms, i.e., cannot be factored $\phi = \varphi \wedge \omega$.

This allows us to to find an ONB A_α for $\Lambda_0^{1,1}$ such that $K(A_\alpha) = \lambda_\alpha A_\alpha$ and for a primitive ϕ:

$$
\begin{aligned}
g\left(\mathrm{Ric}_L(\phi), \bar{\phi}\right) &= \lambda \frac{(p-q)^2}{n} |\phi|^2 + \sum \lambda_\alpha |A_\alpha \phi|^2 \\
&= \sum \lambda_\alpha \left(\frac{(p-q)^2}{n(n-1)} |\phi|^2 + |A_\alpha \phi|^2 \right).
\end{aligned}
$$

If we assume $|\phi|^2 = 1$, then

$$
\sum \left(\frac{(p-q)^2}{n(n-1)} |\phi|^2 + |A_\alpha \phi|^2 \right) = (n+1)(p+q) - 2pq
$$

and

$$
\frac{(p-q)^2}{n(n-1)} |\phi|^2 + |A_\alpha \phi|^2 \leq \left(\frac{(p-q)^2}{n(n-1)} + p + q \right).
$$

Define:

$$C_{p,q} = \frac{(n+1)(p+q) - 2pq}{\frac{(p-q)^2}{n(n-1)} + p + q}$$

Broder-Nienhaus-P-Stanfield-Wink: If K is $C_{p,q}$ positive, then primitive harmonic (p, q)-forms vanish.

- In general: $C_{p,q} \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and is obtained for $\left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right)$-forms.
- $C_{p,0} \geq \frac{p}{n}\left(n^2 - 1\right)$. In particular, $C_{n,0} = n^2 - 1 = \dim \Lambda_0^{1,1}$.
- $C_{p,p} = n + 1 - p$ as in the case of P-Wink.